Math 182

Assignment #5: Approximating Functions by Polynomials

Goals

- to introduce the idea of one function being a good approximation to another.
- To prepare for work on Taylor polynomials and Taylor series.

You have the option of preparing a written record of your investigation with Maple printouts attached or working entirely in a Maple worksheet. If you elect to work entirely within Maple, be sure to become familiar with inserting text regions so that you can provide the required comments to accompany your calculations. Remember, your submission must be a logical and readable report of your investigation.

Procedure

Polynomials can be easily evaluated at any point and their integrals are easy to find. This is not true of many other functions. Thus, it is useful to find polynomials that are good approximations to other functions.

In this lab you will find polynomials that approximate the exponential function. This function is important in mathematics and frequently appears in models of natural phenomena (population growth and radioactive decay, for instance). In these situations you need an easy way to approximate e^x for all values of x, not just for integers and simple fractions. Also, integrals involving the exponential function are important in statistics. For example, $\frac{1}{\sqrt{2\pi}}\int_0^{0.5} e^{-x^2}\,dx$, which calculates the probability of a certain event that follows the "bell curve" of the *normal distribution*, cannot be evaluated in terms of the usual functions of calculus.

You will rely on Maple's ability to evaluate and graph the exponential function in order to determine polynomials that appear to be good approximations to this function. You will also use your polynomial approximations to compute integrals involving the exponential functions.

1. You will begin with a constant function that best approximates e^x near $x = 0$. Why is the graph of $y = 1$ the best constant approximation to the graph of $y = e^x$ near $x = 0$? That is, why would $y = 2$ or $y = -1$ be a worse approximation to $y = e^x$ near $x = 0$? Denote this polynomial approximation of degree zero by p_0.

2. Now you want to add a first degree term to p_0 to find a polynomial of the form $1 + ax$ that best approximates e^x near 0. Use Maple to graph $y = e^x$ and several candidates such as $y = 1 + 0.5x$, $y = 1 + 0.9x$, and $y = 1 + 1.2x$ on the same axes. Keep in mind that you are looking for the value of a so that $1 + ax$ best approximates e^x near $x = 0$. Thus, you should favour a line that follows along the curve $y = e^x$ right at $x = 0$. You may need to change your scale to decide which line is better. [Do NOT restrict your investigation only to the three examples given above.]

   In your report, record which lines you tried and explain the criteria you used in choosing the line that gives the best approximation. Let $p_1(x) = 1 + ax$ denote your choice of the line that best approximates e^x near $x = 0$.

3. Next, find the second degree term bx^2 to add to p_1 to get a quadratic polynomial $p_2(x) = 1 + ax + bx^2$ that best approximates e^x near $x = 0$. Try to get a parabola that follows along the graph of $y = e^x$ as closely as possible on both sides of $x = 0$. Again, record the polynomials you tried and why you finally chose the one you did.

4. Finally, find a third degree term cx^3 to add to p_2 to get a cubic polynomial $p_3(x) = 1 + ax + bx^2 + cx^3$ that best approximates e^x near zero. This may not be so easy; you may have to change scale several times before you see why one polynomial is better than another.

5. Now that you have a polynomial that approximates e^x, try evaluating $p_3(0.5)$ as a computationally simple way of estimating $e^{0.5}$. How close is the polynomial approximation to the value of $e^{0.5}$ as determined by a calculator or Maple? Which is larger? How does the error at other points of the interval $[0, 0.5]$ compare with the error at $x = 0.5$? If you cannot distinguish between the graphs of $y = p_3(x)$ and $y = e^x$, you may want to plot the difference $y = e^x - p_3(x)$ with a greatly magnified scale on the y-axis.

6. Return to the problem of computing a definite integral such as $\int_0^{0.5} e^{-x^2}\,dx$ for which the integrand does not have an antiderivative in terms of elementary functions. Since $p_3(x)$ approximates e^x, you can use $p_3(-x^2)$ to approximate e^{-x^2}.

   (a) Evaluate $\int_0^{0.5} p_3(-x^2)\,dx$ as an approximation to $\int_0^{0.5} e^{-x^2}\,dx$. Note that *you can do this integral by hand*, but go ahead and use Maple if you wish.

   (b) Evaluate the integral $\int_0^{0.5} e^{-x^2}\,dx$ directly using Maple. Be sure to enter the upper limit as 0.5 and not 1/2, or else Maple will not give you a numerical answer. (Go ahead and see how Maple responds to you when you supply the upper limit as 1/2.) Compare your numerical result to the answer from part (a).

7. An analytical method for approximating a function near a point leads to what are known as *Taylor polynomials*. The Taylor polynomial of degree n is determined by matching the values of the polynomial and its first n derivatives with those of the function at a particular point.

   (a) Make a table to compare the values of p_3 and its first three derivatives with the values of e^x and its derivatives, all evaluated at $x = 0$. How close was your polynomial p_3 to being a Taylor polynomial?

   (b) Determine the cubic Taylor polynomial for the exponential function. To do this, adjust the four coefficients so the values of the Taylor polynomial and its first three derivatives match those of e^x at $x = 0$. Plot this polynomial and your polynomial p_3. Compare how close they are to the graph of $y = e^x$ near $x = 0$.