Active learning in High-School mathematics using Interactive Interfaces

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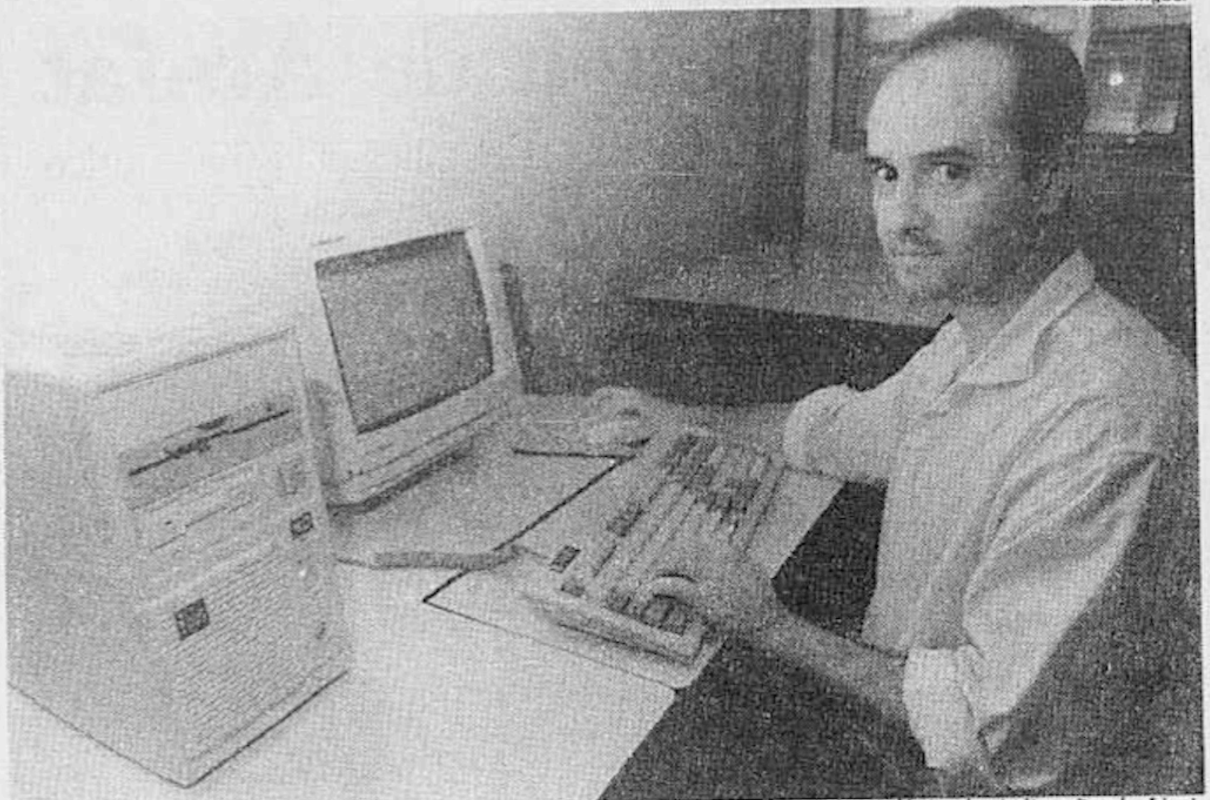
Abstract:

The key idea in this project is to learn through exploration using a web of user-friendly Highly Interactive Graphical Interfaces (HIGI). The HIGIs, structured as trees of interlinked windows, present concepts using a minimal amount of text while maximizing the possibility of visual and analytic exploration. These interfaces run computer algebra software in the background. Assessment tools are integrated into the learning experience within the general conceptual map, the Navigator. This Navigator offers students self-assessment tools and full access to the logical sequencing of course concepts, helping them to identify any gaps in their knowledge and to launch the corresponding learning interfaces. An interactive online set of HIGIS of this kind can be used at school, at home, in distance education, and both individually and in a group.

Computer algebra interfaces for High-School students of "Colegio de Aplicação" (UERJ/1994)

- Pilot experience with high-school students of grades 10 and 11 of the "Colegio de Aplicação of the State University of Rio de Janeiro". Interview of 4/July/1994, from the archives of the "Jornal do Brasil" (at that time a sort of Brazilian version of "The New York Times").

Ismar Ingber



Cheb-Terrab considera excelentes os resultados obtidos junto aos alunos do Colégio de Aplicação da Uerj

Computador ajuda alunos a entender física e matemática

■ Sistemas criados pela Uerj facilitam o ensino no 2º grau

ALICIA IVANISSEVICH

Computadores com tela colorida, imagens em movimento, programas *inteligentes* e teclados para *conversar* com as máquinas estão sendo adotados por professores da Universidade do Estado do Rio de Janeiro (Uerj) para motivar alunos de segundo grau e de graduação a aprender conceitos complexos de matemática e física. Consideradas matérias áridas pela maioria dos estudantes — são responsáveis por alto índice de reprovação —, a matemática e a física passam agora a fazer parte

Laboratório de Física Computacional (Lafic) do Instituto de Física da Uerj resolveram testar *softwares* junto a alunos das primeira e segunda séries do segundo grau do Colégio de Aplicação da Uerj.

"Testamos programas instrutivos e interativos com quatro alunos", aponta Edgardo Cheb-Terrab, coordenador do Lafic. Ele explica que os *softwares* instrutivos são específicos para determinados tipos de problemas, como o estudo do movimento de um pêndulo ou da trajetória de um planeta ao redor do Sol. Já os interati-

Edgardo não dispensa a figura do professor. "O contato humano é imprescindível", afirma. "Mas os programas complementam o aprendizado. As respostas são instantâneas e satisfazem de imediato a curiosidade do aluno. Ele consegue entender todo o procedimento de um cálculo sem se perder pela atenção obsessiva de não errar sinais e contas."

Em quatro meses de trabalho, os resultados foram considerados excelentes pela equipe do Lafic. "Propusemos temas tratados no primeiro ano da universidade, co-

>

Motivation

When we are the average high-school student facing mathematics, we tend to feel

• Bored, fragmentarily taking notes, listening to a teacher for 50 or more minutes
• Anguished because we do not understand some math topics (too many gaps accumulated)
• Powerless because we don't know what to do to understand (don't have any instant-tutor to ask questions and without being judged for having accumulated gaps)
• Stressed by the upcoming exams where the lack of understanding may become evident

Computer algebra environments can help in addressing these issues.

• Be as ***active*** as it can get while ***learning at our own pace***.

• Explore at high speed and ***without feeling judged***. There is space for curiosity with no computational cost.

• Feel ***empowered by success. That leads to understanding.***

• Possibility for *making of* ***learning a social experience.***

Interactive interfaces

Interactive interfaces do not replace the teacher - human learning is an emotional process. A good teacher leading good active learning is a positive experience a student will never forget

Not every computer interface is a valuable resource, at all. It is the set of pedagogical ideas implemented that makes an interface valuable (the same happens with textbooks)

A course on high school mathematics using interactive interfaces - the Edukanet project

– Brazilian and Canadian students/programmers were invited to participate - 7 people worked in the project.

– Some funding provided by the Brazilian Research agency CNPq.

Tasks:

⋆ Develop a framework to develop the interfaces covering the last 3 years of high school mathematics (following the main math textbook used in public schools in Brazil)

⋆ Design documents for the interfaces according to given pedagogical guidelines.

⋆ Create prototypes of Interactive interfaces, running Maple on background, according to design document and specified layout (allow for everybody's input/changes).

The pedagogical guidelines for interactive interfaces

- It is key to address the lack of preparation: provide clear information on the pre-requisites to understand a given topic and help in switching to study the pre-requisites when appropriate (the Navigator).
- Restrict the presentation contents to core knowledge, leave derived knowledge to the exercises.
- Minimize the amount of text, maximize the possibility of algebraic exploration with visualization.
- Minimize the requirements to understand a new concept (Example: explain derivatives without requiring understanding of physics)
- Clearly distinguish "definitions" (as words of a dictionary - provide links to them) from "mathematical concepts" (the topics being presented).

The Math-contents design documents for each chapter

Example: complex numbers

Complex numbers

Part 1) Algebra

Properties and operations: addition, subtraction, multiplication, exponents, conjugates, division (using conjugates and linear systems);

Window 1 - The Imaginary Unit

Panel Definitions:

The imaginary unit is $i = \sqrt{-1}$, such that $i^2 = -1$.
The imaginary unit is a new concept: a number whose square is negative. We will see that the properties of exponents, for complex numbers, are a generalization of the properties of exponents of real numbers.

By convention, we use the variable z to indicate a complex number, $z = x + y\,i$, where
The real part of z is $a \in \Re$, that is, $a = \Re(z)$
The imaginary part of z is $b \in \Re$, the coefficient of i, that is, $b = \Im(z)$.

Daniel, it would probably look nice if you'd make a box with $a = \Re(z)$ and $b = \Im(z)$, with the explanations below the box.

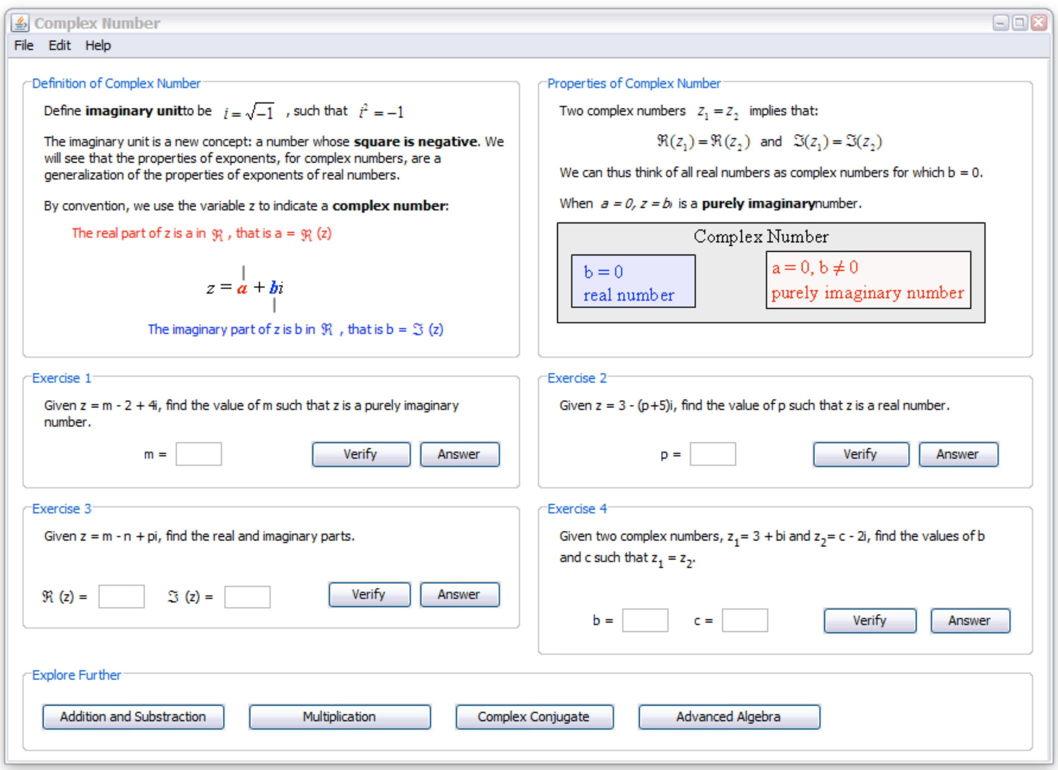
We can thus think of all real numbers as complex numbers for which $b = 0$.
When $a = 0$, $z = b\,i$ is a purely imaginary number.

Panel Exercises:

1. Given $z = m - 2 + 4\,i$, find the value of m such that z is a purely imaginary number; m = ...
2.and 3. Two more examples for practicing the real/imaginary part of a number.
4. Given two complex numbers, $z_1 = 3 + b\,i$ and $z_2 = c - 2\,i$, find the values of b and c such that $z_1 = z_2$. Message: Note that $z_1 = z_2$ implies that $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$.

Notes:

- the exercises need buttons alongside for correction, etc.
- the message should appear in the box with the correction, whether the exercise was done correctly or not.



Window 2 - Addition and Subtraction of Complex Numbers

Panel "Addition"

When we add two complex numbers,

$$> z_1 := a + b\,I;$$
$$z_2 := c + d\,I$$

$$z_1 := a + \mathrm{I}\, b$$
$$z_2 := c + \mathrm{I}\, d \qquad \textbf{(3.1.2.1.1.2.1)}$$

> $z_3 := z_1 + z_2$

$$z_3 := a + \mathrm{I}\, b + c + \mathrm{I}\, d \qquad \textbf{(3.1.2.1.1.2.2)}$$

what do you notice about the real part of z_3 and the imaginary part of z_3?

$\Re(z_3) = \ldots$

$\Im(z_3) = \ldots$

When these are filled out, there should appear a message showing that $a + c = \Re(z_1) + \Re(z_2)$ and $b + d = \Im(z_1) + \Im(z_2)$.

Panel "Subtraction"

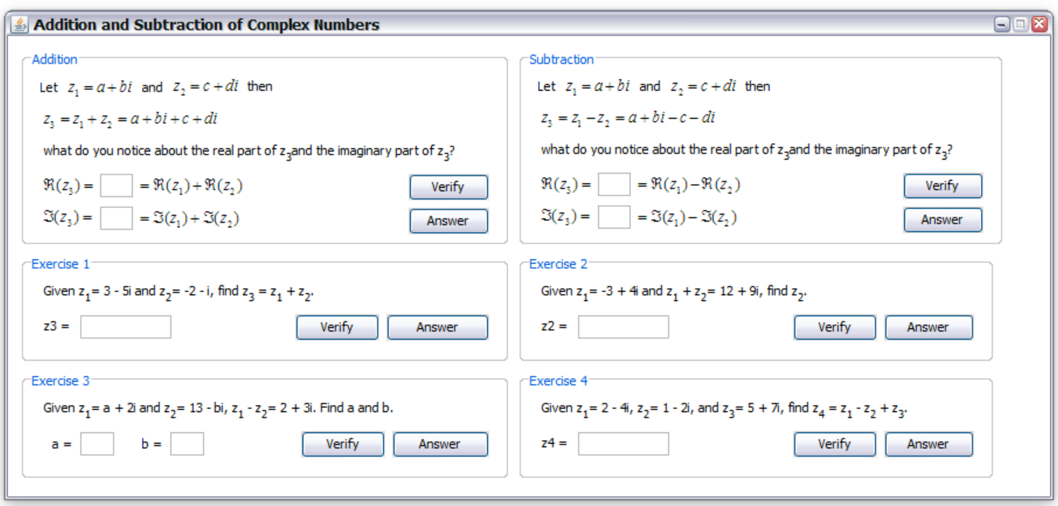
When subtracting, we need to make sure to use the distributive rule: $z_1 - z_2 = a + b\,i - (c + d\,i)$:

> $z_1 - z_2$

$$a + \mathrm{I}\, b - c - \mathrm{I}\, d \qquad \textbf{(3.1.2.1.1.2.3)}$$

Panel "Exercises"

>



Window 3 - Multiplication of Complex Numbers

Panel "Product"

Let us multiply two complex numbers:

> $z_1 := a + b\,\mathrm{I};$
> $z_2 := c + d\,\mathrm{I}$

$$z_1 := a + \mathrm{I}\, b$$
$$z_2 := c + \mathrm{I}\, d \qquad \textbf{(3.1.2.1.1.3.1)}$$

Daniel, the idea is for you to show the product with arrows pointing towards where each term came from.

> $expand(z_1 z_2)$

$$a c + \mathrm{I}\, a d + \mathrm{I}\, b c - b d \tag{3.1.2.1.1.3.2}$$

and stress the fact that $i^2 = -1$.
Then, ask the student to write the imaginary part and the real part of the result.

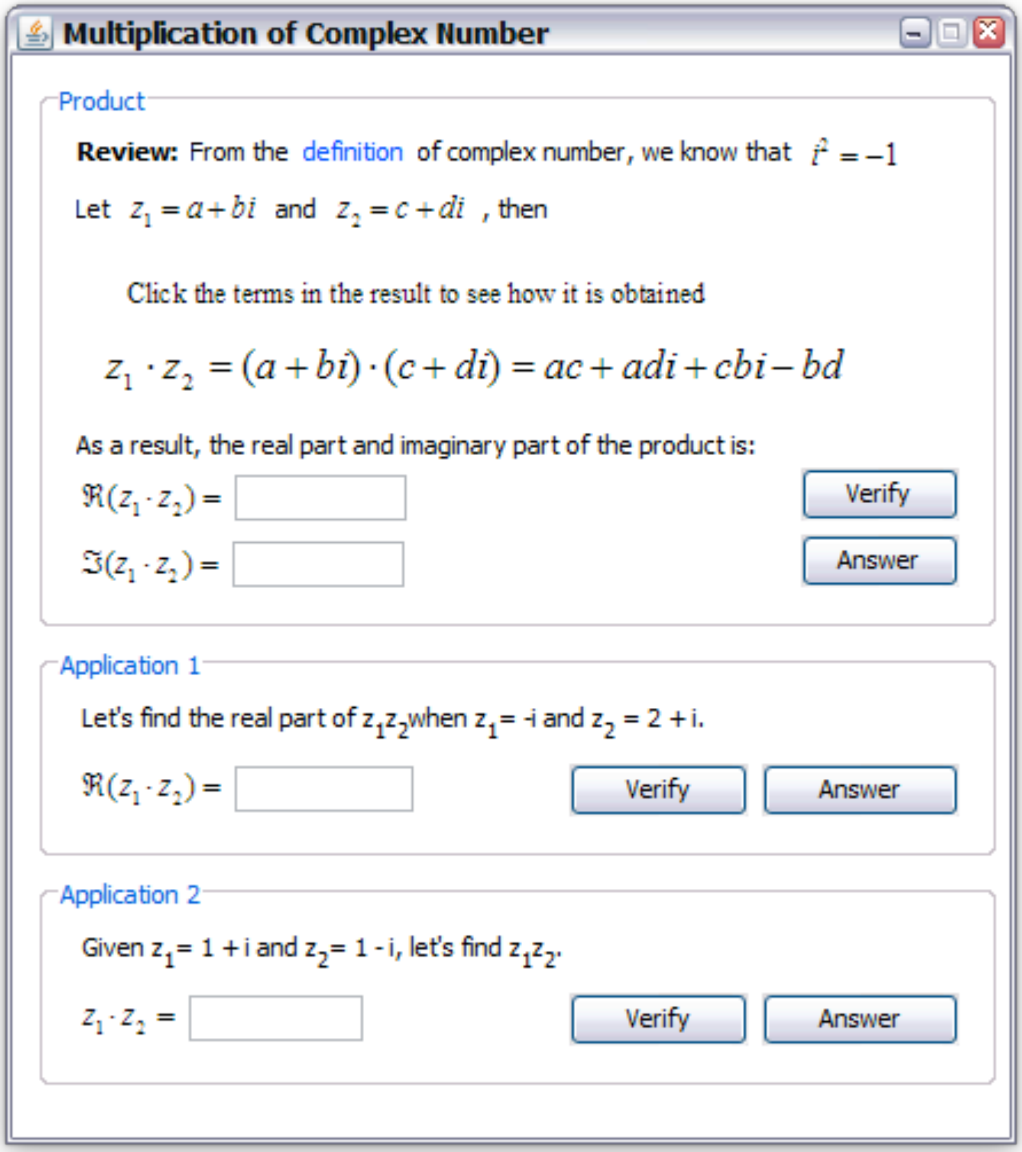
Panel "Aplications"
1. Let us find the real part of $z_1 z_2$ when $z_1 = -i$, $z_2 = 2 + i$.

> $z_1 := -i :$
> $z_2 := 2 + i :$

$\Re(-i\,(2 + i)) = \ldots$
(Buttons Correct and Calculate...)

2. Given $z_1 = 1 + i$ and $z_2 = 1 - i$, let's find $z_1 z_2$.

$z_1 z_2 = \ldots$
(Buttons Correct and Calculate...)
Observe that z_1 and z_2 differ only in the sign of their imaginary parts. These numbers are called **complex conjugate**. You can verify [link to the window of Complex Conjugates] that the product of two complex conjugates is always a real number.



Window 4 - Complex Conjugates

Panel "Using Complex Conjugates"

1. Write a complex number z and its conjugate $\overline{z}$ and find the product:

Daniel, you could put three boxes here, one for z = ... ,the other for $\overline{z}$ = ... , and the last for $z\,\overline{z}$ = ...

Button: CALCULATE which does the multiplication step by step, if the person typed in the number and its conjugate correctly, or gives an example otherwise; and a button CORRECT.

2. Let $z_1 = \dfrac{1}{1-I}$. Just like when we rationalize a fraction to avoid square roots in denominators, we will also avoid imaginary numbers in denominators. Since a complex number multiplied by its conjugate gives a real result, we rationalize z_1 using

the conjugate of its denominator:

$$z_1 = \frac{1}{1 - I} * \frac{1 + I}{1 + I} = \ldots$$

(Buttons Calculate and Correct)
Note that now z_1 is in the form a + bi.

Complex conjugates are useful, for instance, in the division between complex numbers.
Daniel, the word division should link to the next window.

Window 5 - Division of Complex Numbers

Panel "Division"

Let's take two complex numbers $z_1 = a + b\,i$, $z_2 = c + d\,i$ and find $\frac{z_1}{z_2}$.

We'll have $\frac{z_1}{z_2} = \frac{a + b\,i}{c + d\,i}$ but how do we write this as a complex number, that is, in the form $x + y\,i$? We use the complex conjugate of the denominator:

$$\frac{z_1}{z_2} = \frac{a + b\,i}{c + d\,i} = \frac{a + b\,i}{c + d\,i}\,\frac{c - d\,i}{c - d\,i}$$

The result is that $\frac{z_1}{z_2} =$ ....
[button Calculate shows the multiplication step-by-step, button Correct]

Panel "Exercises"

Consider the complex numbers $z_1 = 3 + i$, $z_2 = 4 - 2\,i$.

1. Calculate $\frac{z_1}{z_2}$
2. Calculate $\frac{z_1 + z_2}{i\,z_1}$

Window 6 - Exercises

[Daniel, here go some exercises where the student can apply all that has been learned so far.

Part 2) Advanced Algebra

Module, phase and exponential representation

Window 1 - Graphic Representation of a Complex Number

Panel "The Gauss Plane"
A complex number z can be represented graphically on a Cartesian plane where the horizontal axis represents $\Re(z)$ and the vertical axis represents $\Im(z)$:
[Daniel, the idea is to have a graph showing a point labeled $z = a + b\,i$ with dotted lines dropping to the horizontal axis showing $a = \Re(z)$ and to the vertical axis showing $b = \Im(z)$]
The plane above is also called the Gauss plane, and its axes are called real axis and imaginary axis.
Panel "Exercises"
[Daniel, here go a couple of simple exercises to make sure the student knows how to write the complex number corresponding to a point on the Gauss plane and on which axis a pure imaginary number would lie]

Window 2 - Module and Phase of a Complex Number

Panel "Module"

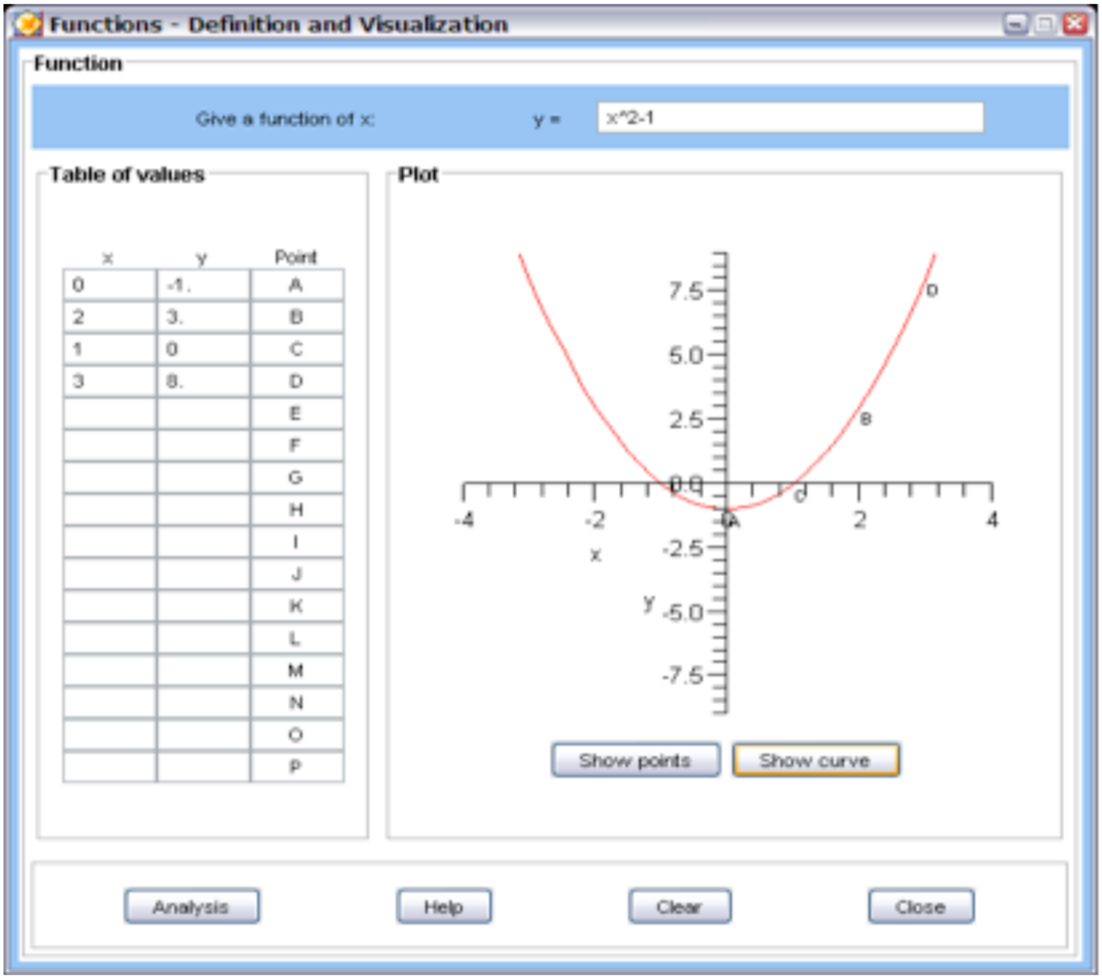
>

Each math topic: a interactive interrelated interfaces (windows)

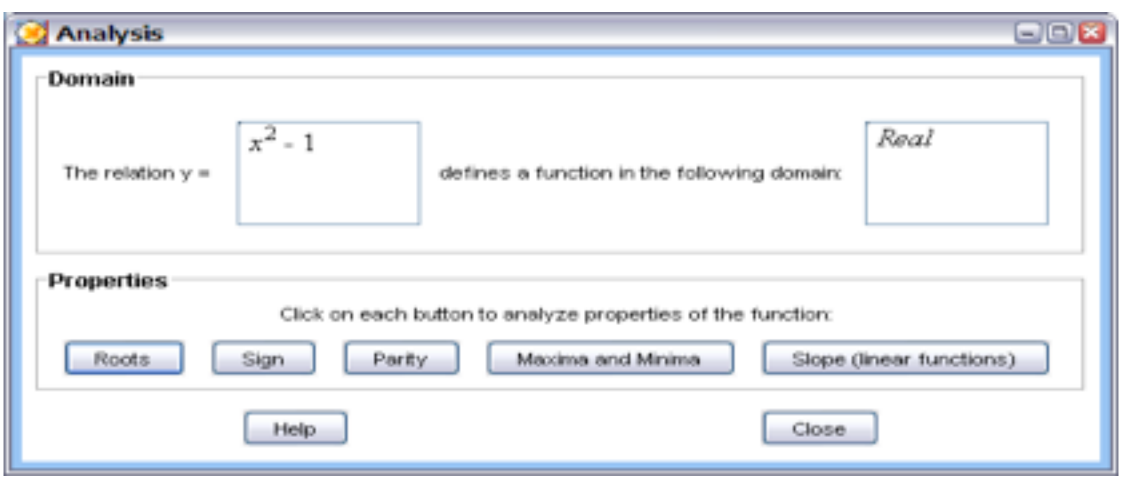
For each topic of high-school mathematics (chapter of a textbook), develop a ***tree of interactive interfaces (applets) related to the topic*** (main) and subtopics

Example: *Functions*

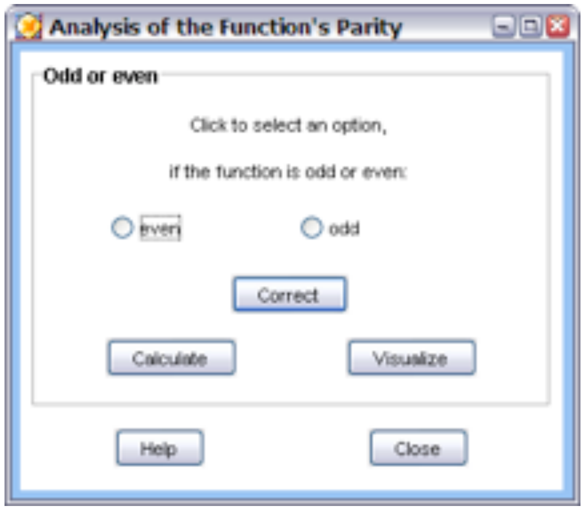
• Main window



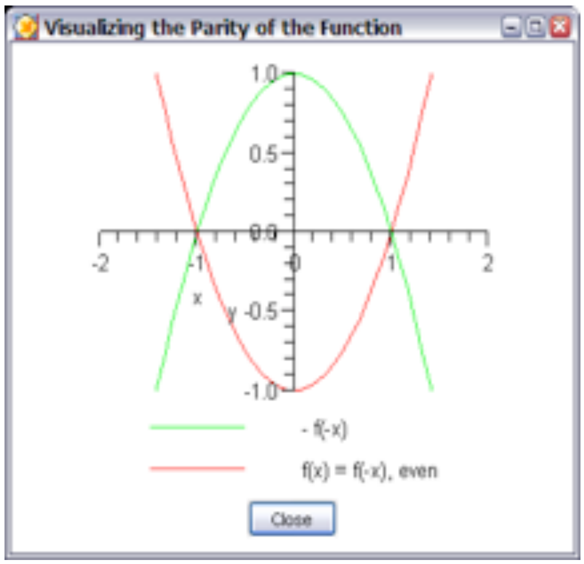
• Analysis window



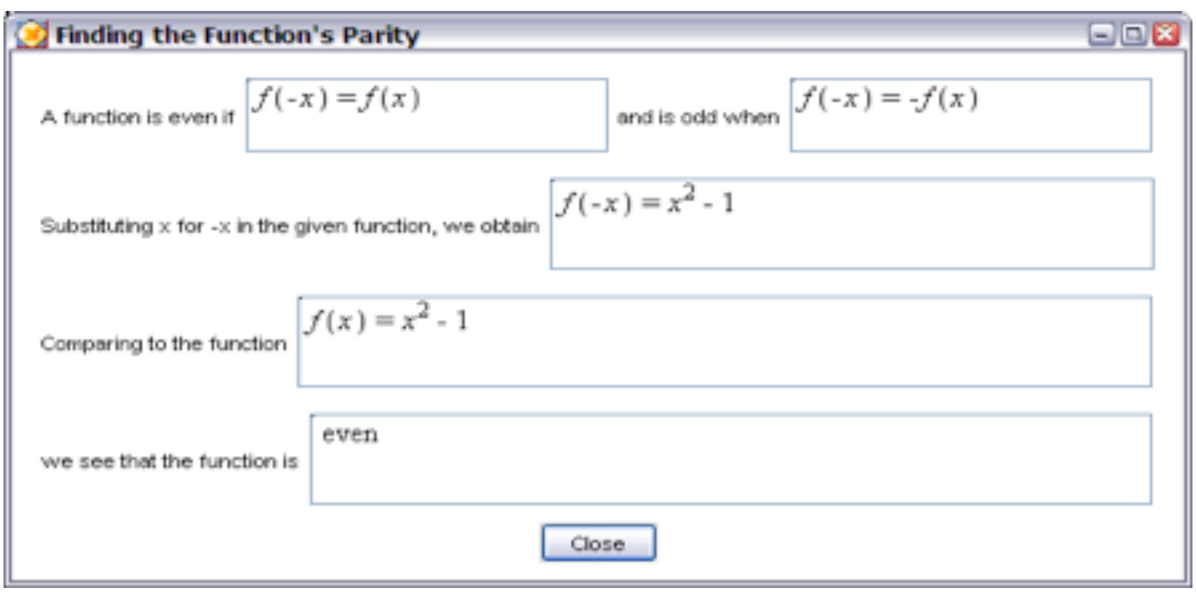
• Parity window



• Visualization of function's parity



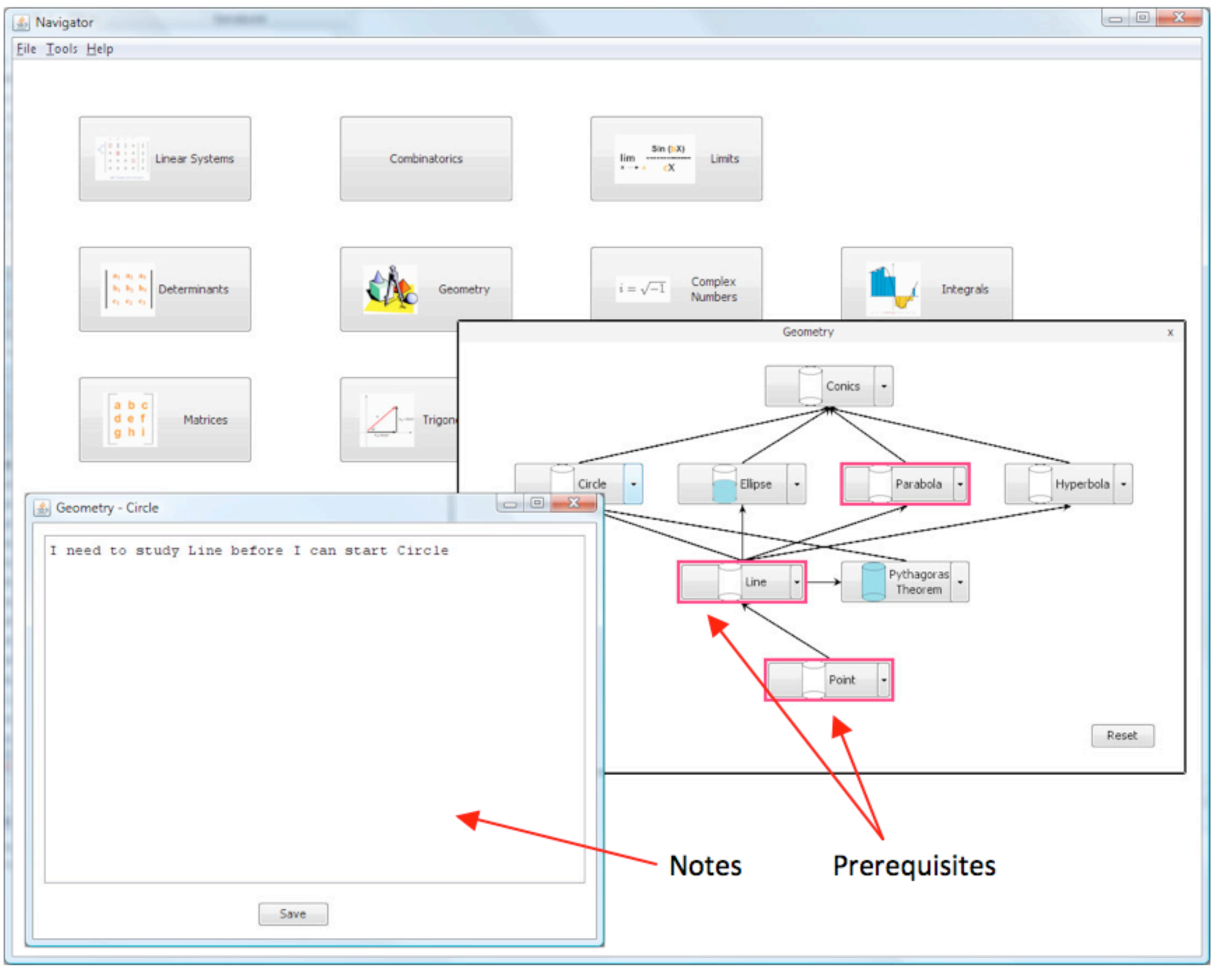
• Step-by-Step solution window



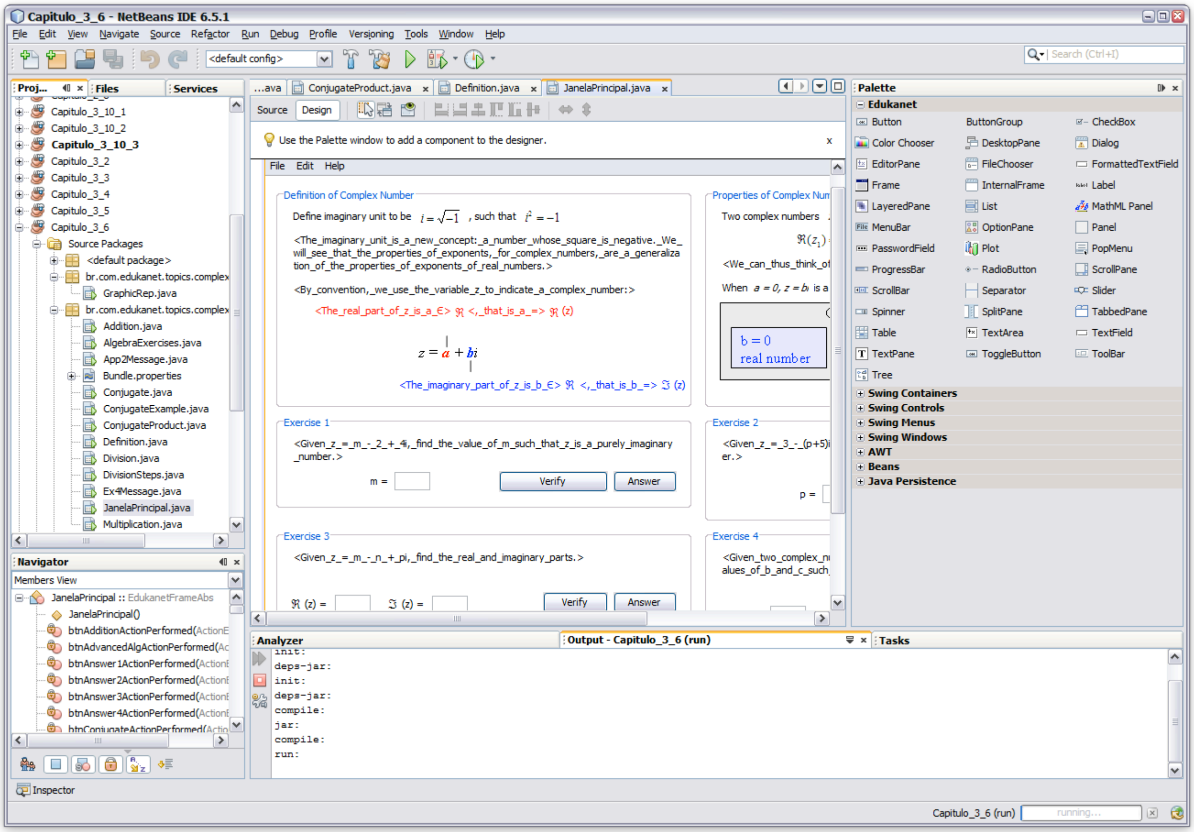
The Navigator: a window with a tile per math topic

- Click the topic-tile to launch a smaller window, topic-specific, map of interrelated sub-topic tiles, that indicates the logical sequence for the sub-topics, and from where one could launch the corresponding sub-topic interactive interface.

- This topic-specific smaller window allows for ***identifying the pre-requisites and gaps in understanding***, launching the corresponding interfaces to fill the gaps, and tracking the level of familiarity with a topic.



▼ ***The framework to create the interfaces: a version of NetBeans on steroids …***



Complementary classroom activity on a computer algebra worksheet

This course is organized as a guided experience, 2 hours per day during five days, on learning the basics of the Maple language, and on using it to formulate algebraic computations we do with paper and pencil in high school and 1st year of undergraduate science courses.

Explore. Having success doesn't matter, using your curiosity as a compass does - things can be done in so many different ways. Have full permission to fail. Share your insights. All questions are valid even if to the side. Computer algebra can transform the learning of mathematics into interesting understanding, success and fun.

1. Arithmetic operations and elementary functions

Operators :	+, -, *, /, ^
Functions	exp, ln, sin, cos, tan, csc, sec, cot, arcsin, arccos, arctan, arccsc, arcsec, arccot. For the hyperbolic functions put an h at the end as in sinh, arctanh, etc.
Manipulation commands	Related to numerical evaluation: evalf, Digits. The complex components: Re, Im, conjugate, abs, argument Related to functions: series, convert (any function to any other one when possible), FunctionAdvisor Related to plotting: plot, plot3d, plots:-plotcompare

Examples

Blank spaces mean multiplication.

Function application is represented with rounded parenthesis (), as in $f(x)$.

Indexation, as used in tensors, is represented with squared brackets [], as in A[mu] displayed as A_{mu}.

Numerical approximation is obtained applying *evalf*

> $restart; interface(imaginaryunit = i):$

> $4 + 5\,i$

$$4 + 5\,\mathrm{i} \qquad (3.2.1.1.1)$$

> $\mathrm{Re}(\mathbf{(3.2.1.1.1)})$

$$4 \qquad (3.2.1.1.2)$$

> $conjugate(\mathbf{(3.2.1.1.1)})$

$$4 - 5\,\mathrm{i} \qquad (3.2.1.1.3)$$

> $evalf(\mathrm{Pi})$

$$3.141592654 \qquad (3.2.1.1.4)$$

> $Digits$

$$10 \qquad (3.2.1.1.5)$$

> $evalf[50](\mathrm{Pi})$

$$3.1415926535897932384626433832795028841971693993751 \qquad (3.2.1.1.6)$$

> $FunctionAdvisor(\;)$

```
The usage is as follows:
    > FunctionAdvisor( topic, function, ... );
where 'topic' indicates the subject on which advice is
required, 'function' is the name of a Maple function, and
'...' represents possible additional input depending on
the 'topic' chosen. To list the possible topics:
    > FunctionAdvisor( topics );
A short form usage,
    > FunctionAdvisor( function );
with just the name of the function is also available and
displays a summary of information about the function.
```

> $FunctionAdvisor(topic)$

```
* Partial match of "topic" against topic "topics".
The topics on which information is available are:
```

[*DE*, *analytic_extension*, *asymptotic_expansion*, *branch_cuts*, *branch_points*, *calling_sequence*, *class_members*, *classify_function*, *definition*, *describe*, *differentiation_rule*, *function_classes*, *identities*, *integral_form*, *known_functions*, *periodicity*, *plot*, *relate*, *required_assumptions*, *series*, *singularities*, *special_values*, *specialize*, *sum_form*, *symmetries*, *synonyms*, *table*] **(3.2.1.1.7)**

> $FunctionAdvisor(classes)$

[*trig*, *trigh*, *arctrig*, *arctrigh*, *elementary*, *GAMMA_related*, *Psi_related*, *Kelvin*, *Airy*, *Hankel*, *Bessel_related*, *0F1*, *orthogonal_polynomials*, *Ei_related*, **(3.2.1.1.8)** *erf_related*, *Kummer*, *Whittaker*, *Cylinder*, *1F1*, *Elliptic_related*, *Legendre*, *Chebyshev*, *2F1*, *Lommel*, *Struve_related*, *hypergeometric*, *Jacobi_related*, *InverseJacobi_related*, *Elliptic_doubly_periodic*, *Weierstrass_related*, *Zeta_related*, *complex_components*, *piecewise_related*, *Other*, *Bell*, *Heun*, *Appell*, *trigall*, *arctrigall*, *integral_transforms*]

> *FunctionAdvisor*(*ele*)

```
* Partial match of "ele" against topic "elementary".
The 26 functions in the "elementary" class are:
```

[arccos, arccosh, arccot, arccoth, arccsc, arccsch, arcsec, arcsech, arcsin, arcsinh, arctan, arctanh, cos, cosh, cot, coth, csc, csch, exp, ln, sec, sech, sin, sinh, tan, tanh] **(3.2.1.1.9)**

> *FunctionAdvisor*(*identities*, sin)

$$\left[\sin(\arcsin(z)) = z, \sin(z) = -\sin(-z), \sin(z) = 2\sin\left(\frac{z}{2}\right)\cos\left(\frac{z}{2}\right), \sin(z) = \frac{1}{\csc(z)}, \sin(z) = \frac{2\tan\left(\frac{z}{2}\right)}{1+\tan\left(\frac{z}{2}\right)^2}, \sin(z) = -\frac{\mathrm{i}}{2}\left(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}\right), \sin(z)^2 = 1-\cos(z)^2, \sin(z)^2 = \frac{1}{2} - \frac{\cos(2z)}{2}\right] \quad (3.2.1.1.10)$$

> *FunctionAdvisor*(*display*, ln)

▼ ln

- ▶ **describe**
- ▶ **definition**
- ▶ **classify function**
- ▶ **symmetries**
- ▶ **periodicity**
- ▶ **plot**
- ▶ **singularities**
- ▶ **branch points**
- ▶ **branch cuts**
- ▶ **special values**
- ▶ **identities**

- **sum form**
- **series**
- **asymptotic expansion**
- **integral form**
- **differentiation rule**
- **DE**

> $\cos(x) + \mathrm{i}\sin(x)$

$$\cos(x) + \mathrm{i}\sin(x) \qquad \textbf{(3.2.1.1.11)}$$

> $convert(\%, \exp)$

$$e^{\mathrm{i}\,x} \qquad \textbf{(3.2.1.1.12)}$$

> $convert(\%, trig)$

$$\cos(x) + \mathrm{i}\sin(x) \qquad \textbf{(3.2.1.1.13)}$$

> $FunctionAdvisor(relate, \arcsin, \ln)$

$$\arcsin(z) = -\mathrm{i}\ln\left(\mathrm{i}\,z + \sqrt{-z^2+1}\right) \qquad \textbf{(3.2.1.1.14)}$$

> $FunctionAdvisor(specialize, \arcsin)$

$$\left[\arcsin(z) = z\,F_1\left(\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, 0, z^2\right), \textit{with no restrictions on } (z)\right], \qquad \textbf{(3.2.1.1.15)}$$

$$\left[\arcsin(z) = z\,F_2\left(\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 0, z^2\right), \textit{with no restrictions on } (z)\right],$$

$$\left[\arcsin(z) = z\,F_3\left(0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, 0, z^2\right), \textit{with no restrictions on } (z)\right],$$

$$\left[\arcsin(z) = z\,F_4\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 0, z^2\right), \textit{with no restrictions on } (z)\right],$$

$$\left[\arcsin(z) = \frac{z\,HC\left(0, \frac{1}{2}, 0, 0, \frac{1}{4}, \frac{z^2}{z^2-1}\right)}{\sqrt{-z^2+1}},\right.$$

$$\left.\textit{with no restrictions on } (z)\right], \left[\arcsin(z) = z\,HG\left(0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, z^2\right),\right.$$

$$\left.\textit{with no restrictions on } (z)\right], \left[\arcsin(z) = \frac{\pi}{2}\right.$$

$$\left. + \frac{am^{-1}(\operatorname{arcsec}(z)|1)\ (z-1)}{\sqrt{-(z-1)^2}}, \Re(z) \in (0, \pi)\right], \left[\arcsin(z)\right.$$

$$= \frac{z\,\pi\,P_{-\frac{1}{2}}^{\left(\frac{1}{2},\,-\frac{1}{2}\right)}\left(-2\,z^2+1\right)}{2}, \textit{ with no restrictions on } (z)\Bigg], \Bigg[\arcsin(z)$$

$$= \frac{z\,\sqrt{\pi}\,\left(-2\,z^2+2\right)^{1/4}\,P_{-\frac{1}{2}}^{-\frac{1}{2}}\left(-2\,z^2+1\right)}{2\,\left(-2\,z^2\right)^{1/4}},$$

$$\textit{with no restrictions on } (z)\Bigg], \Bigg[\arcsin(z) = \frac{z\,G_{2,\,2}^{1,\,2}\left(-z^2\,\middle|\,\begin{matrix}\frac{1}{2},\,\frac{1}{2}\\ 0,-\frac{1}{2}\end{matrix}\right)}{2\,\sqrt{\pi}},$$

$$\textit{with no restrictions on } (z)\Bigg], \Bigg[\arcsin(z) = \frac{\pi}{2} - \arccos(z),$$

$$\textit{with no restrictions on } (z)\Bigg], \Bigg[\arcsin(z) = \frac{\pi}{2} + \frac{\operatorname{arccosh}(z)\,(z-1)}{\sqrt{-(z-1)^2}},$$

$$\textit{with no restrictions on } (z)\Bigg], \Bigg[\arcsin(z) = \pi$$

$$-2\operatorname{arccot}\left(\frac{z}{1+\sqrt{-z^2+1}}\right), \textit{ with no restrictions on } (z)\Bigg],$$

$$\Bigg[\arcsin(z) = \frac{1}{\mathrm{i}\,z+\sqrt{-z^2+1}+1}\Bigg(\Big(2\,\mathrm{i}\sqrt{-z^2+1}+2\,\mathrm{i}$$

$$-2\,z\Big)\operatorname{arccoth}\left(\frac{-\mathrm{i}\,z}{1+\sqrt{-z^2+1}}\right)+\mathrm{i}\,\pi\,\Big(1$$

$$+\sqrt{-z^2+1}\Big)\sqrt{-\left(\frac{\mathrm{i}\,z}{1+\sqrt{-z^2+1}}+1\right)^2}\Bigg),$$

$$\textit{with no restrictions on } (z)\Bigg], \Bigg[\arcsin(z) = \operatorname{arccsc}\left(\frac{1}{z}\right),$$

$$\textit{with no restrictions on } (z)\Bigg], \Bigg[\arcsin(z) = \mathrm{i}\,\operatorname{arccsch}\left(\frac{\mathrm{i}}{z}\right),$$

$$\textit{with no restrictions on } (z) \Big], \Big[\arcsin(z) = \frac{\pi}{2} - \operatorname{arcsec}\left(\frac{1}{z}\right),$$

$$\textit{with no restrictions on } (z) \Big], \Bigg[\arcsin(z) = \frac{\pi}{2}$$

$$+ \frac{\operatorname{arcsech}\left(\frac{1}{z}\right)(-1+z)}{\sqrt{-\left(\frac{1}{z}-1\right)^2 z^2}}, \textit{with no restrictions on } (z) \Bigg], [\arcsin(z)$$

$$= -\mathrm{i}\,\operatorname{arcsinh}(\mathrm{i}\,z), \textit{with no restrictions on } (z)], \Big[\arcsin(z)$$

$$= 2\arctan\left(\frac{z}{1+\sqrt{-z^2+1}}\right), \textit{with no restrictions on } (z)\Big], \Big[\arcsin(z) =$$

$$-2\,\mathrm{i}\,\operatorname{arctanh}\left(\frac{\mathrm{i}\,z}{1+\sqrt{-z^2+1}}\right), \textit{with no restrictions on } (z)\Big], \Big[\arcsin(z)$$

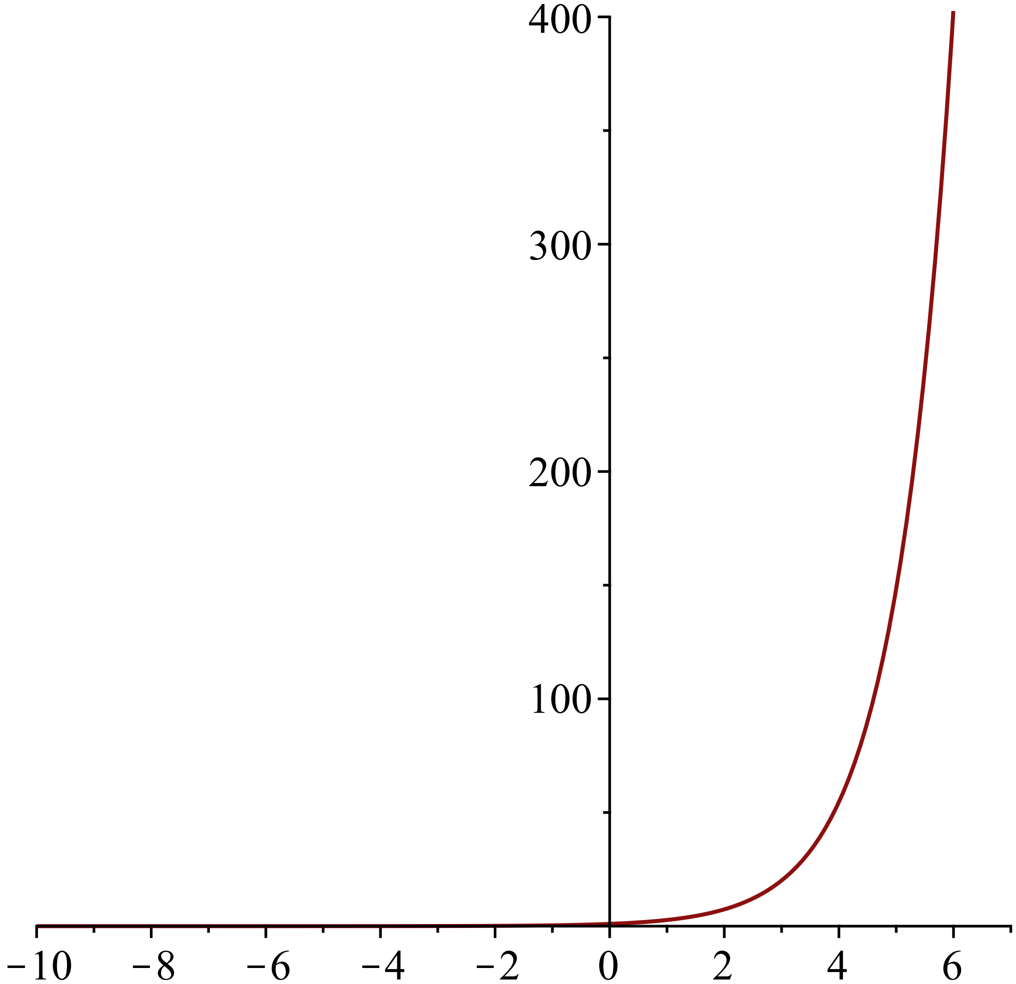
$$= z\;{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right), \textit{with no restrictions on } (z)\Big], \Big[\arcsin(z)$$

$$= -\mathrm{i}\ln\left(\mathrm{i}\,z + \sqrt{-z^2+1}\right), \textit{with no restrictions on } (z)\Big]$$

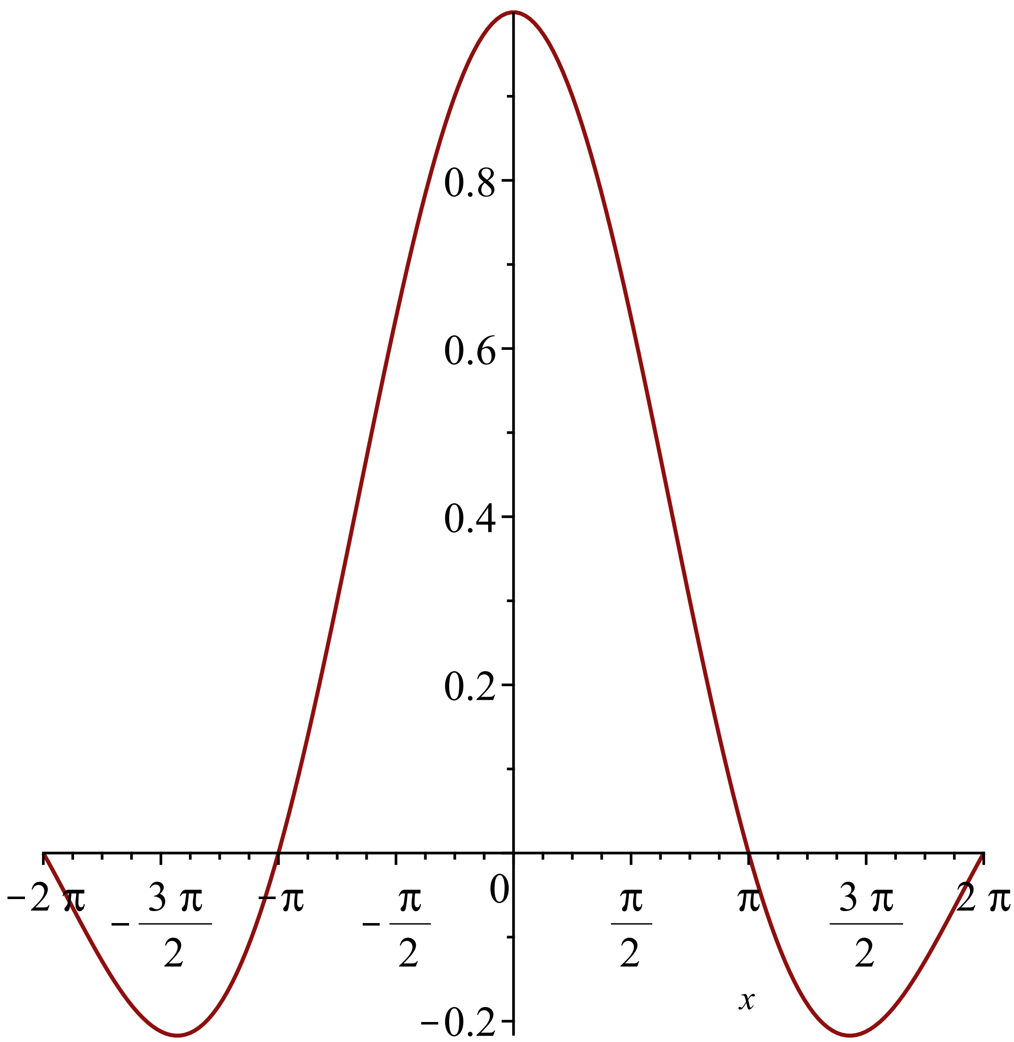
Plotting

2D plotting

> $plot(\exp)$



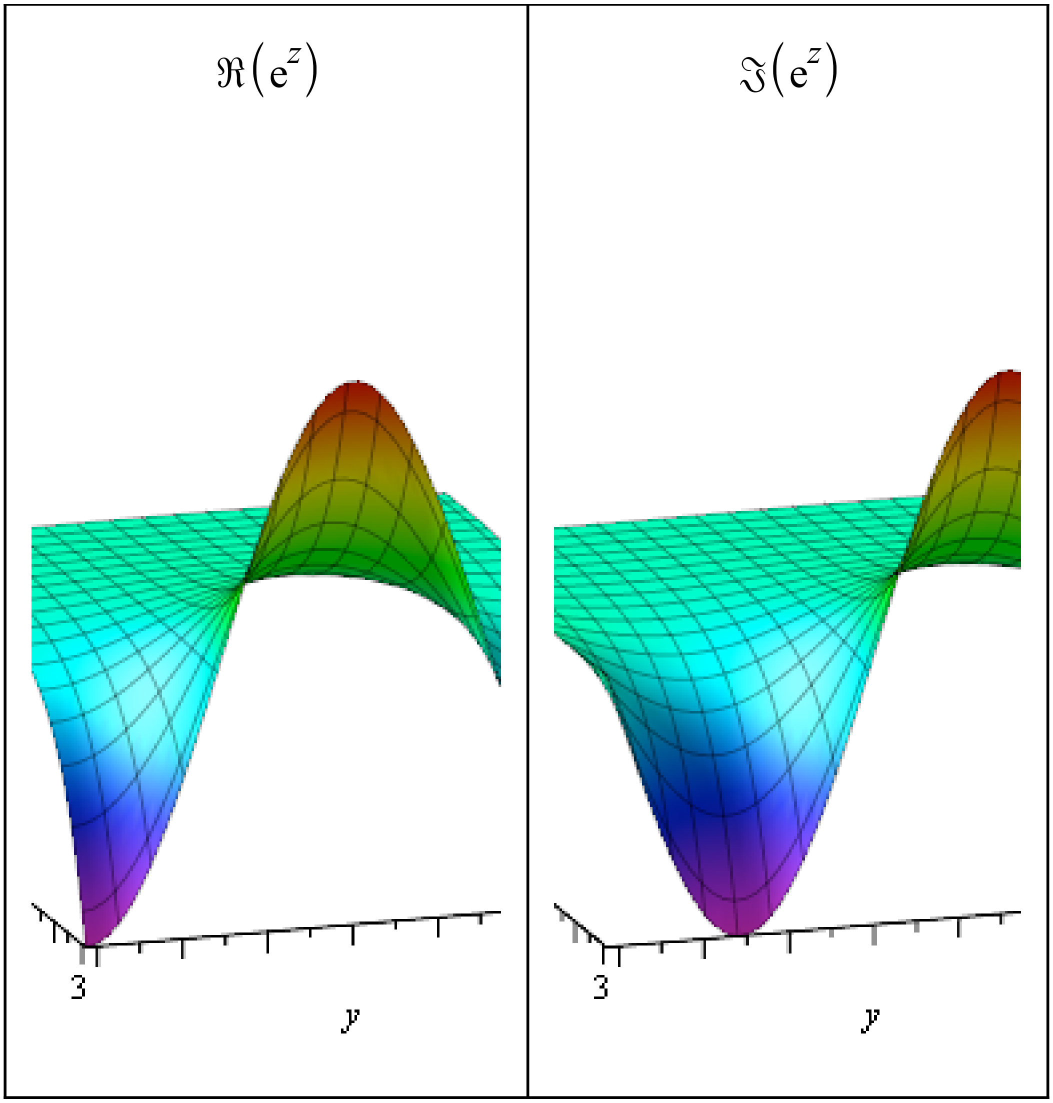
> $plot\left(\frac{\sin(x)}{x}\right)$



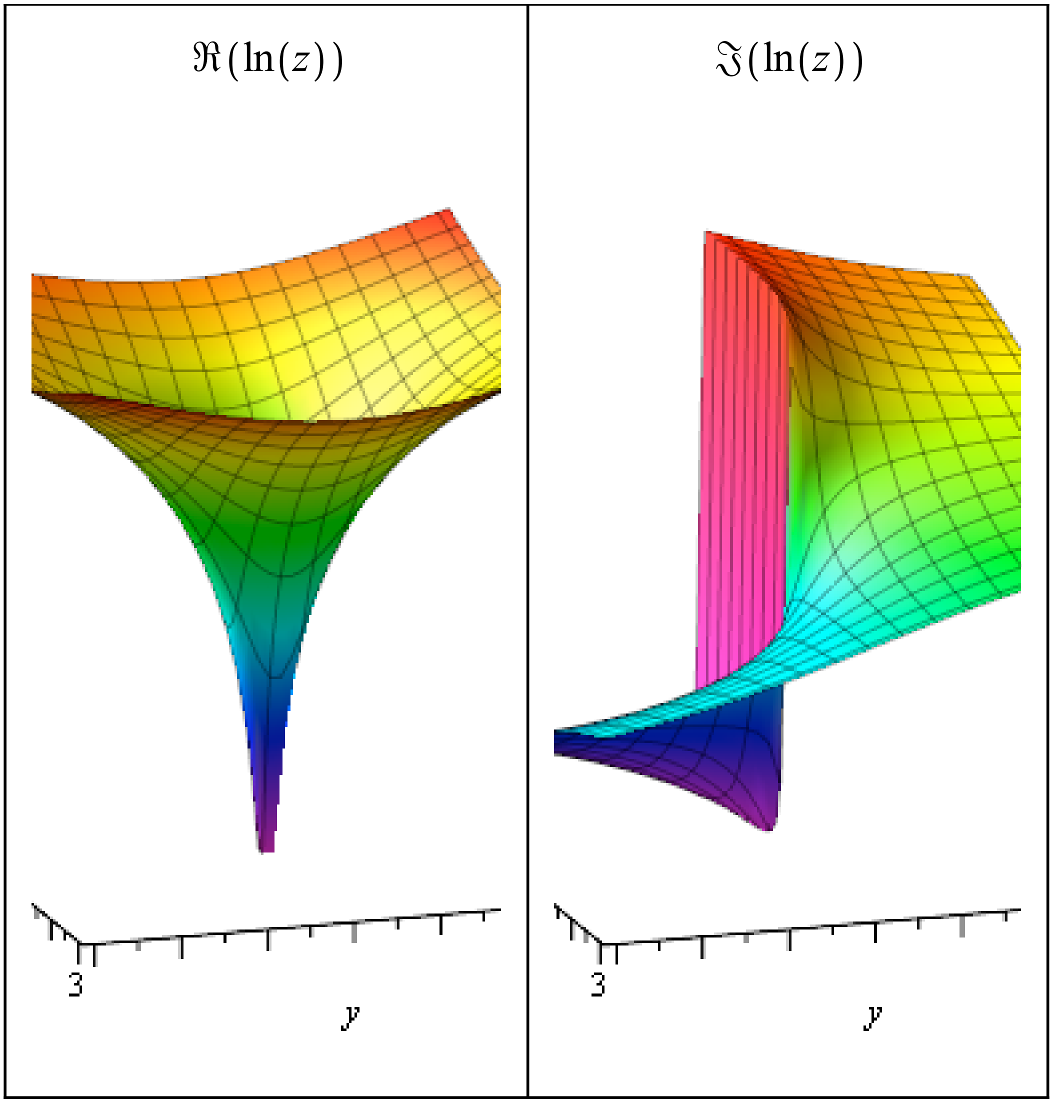
Complex algebraic expressions $F(z)$ of a complex variable z can be represented by two 3D plots: the value of $\Re(F(z))$ and $\Im(F(z))$ (so the real or imaginary parts of the expression) on the vertical axis and the real and imaginary parts of the variable z on the two horizontal axes (PlotExpression uses plots[plotcompare] with the option *expression_plot*)

> $PlotExpression := f \rightarrow plots:\text{-}plotcompare(f, 0, \_rest, 'expression\_plot', 5):$

> $PlotExpression(\exp(z), scale\_range = \text{Pi}, 5)$



> *PlotExpression*(ln(*z*), *scale_range* = Pi, 5)



So ln is a multivalued function with a cut over

> *FunctionAdvisor*(*branch_cuts*, ln)

$$[\ln(z), z < 0] \qquad \textbf{(3.2.1.1.2.1)}$$

> epsilon := $10.0^{(-6)}$

$$\epsilon := 1.000000000\ 10^{-6} \qquad \textbf{(3.2.1.1.2.2)}$$

> $\ln(-i\,\epsilon)$

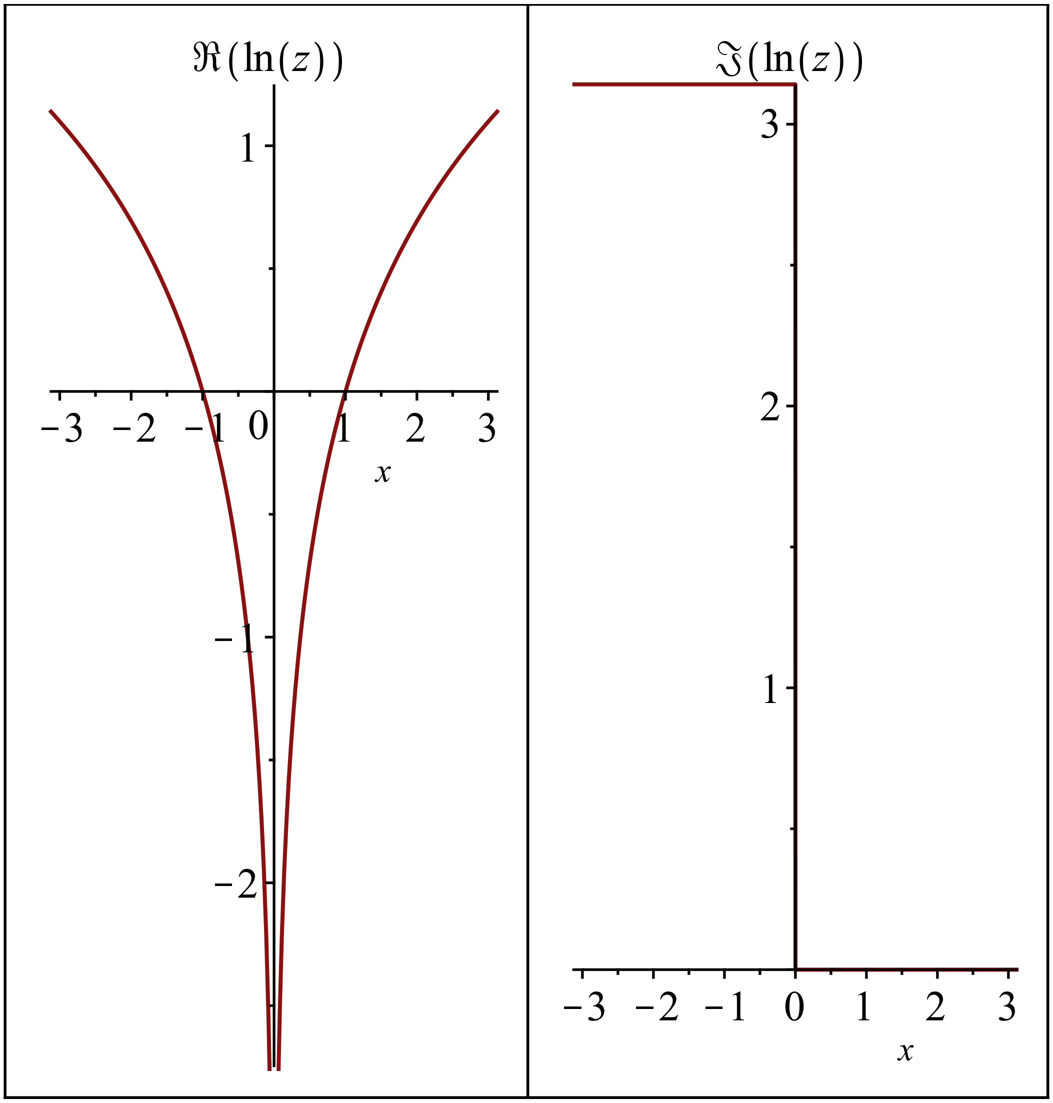
$$-13.81551056 - 1.570796327\,\mathrm{i} \qquad \textbf{(3.2.1.1.2.3)}$$

> $\ln(i\,\epsilon)$

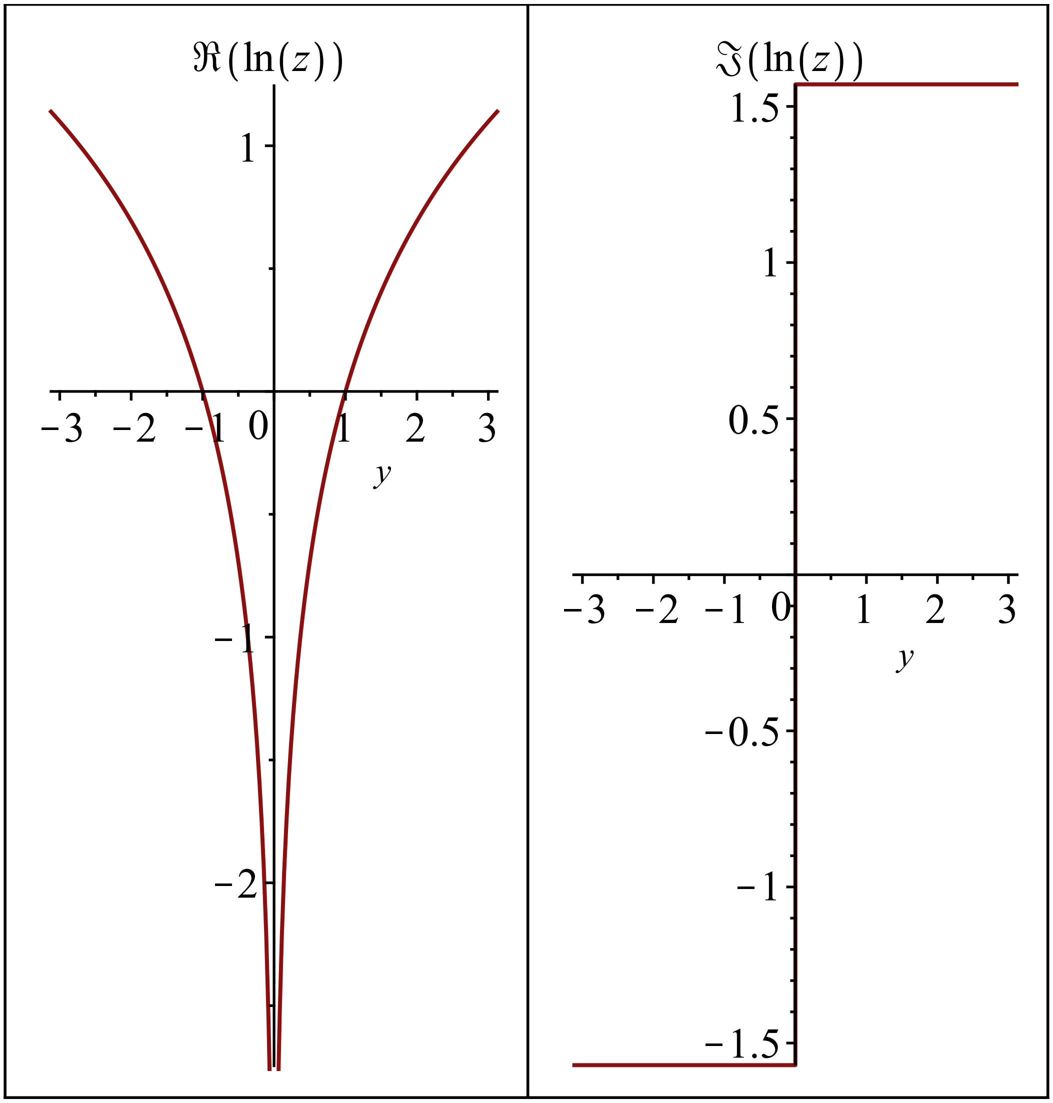
$$-13.81551056 + 1.570796327\,\mathrm{i} \qquad \textbf{(3.2.1.1.2.4)}$$

If you know in advance that z is real, or imaginary then the two 3D plots transform into two 2D plots:

> *PlotExpression*(ln(*z*), *scale_range* = Pi, 5) assuming *z* :: *real*



> *PlotExpression*(ln(*z*), *scale_range* = Pi, 5) assuming *z* :: *imaginary*



>

Exercises

Choose one exercise, try to solve it in up to 10 minutes. If there is time, move to next problem. Or, feel free to use the time to explore the help pages about any related topic more of your interest.

1. Use the convert command to express the functions of the following groups in terms of each other

   a. [exp, sin, cos, tan, sec, csc, cot]

   b. [ln, arcsin, arccos, arctan arcsec, arccsc, arcot]

   c. Advanced: choose a couple of the relations between functions say A = B obtained and verify that A and B have the same series expansion

Solution

2. *Plot* the *sin* function between $-$ pi and pi, then:

1. Click the *plot* to select it
2. Go to the menu Plot –> Probe Info –> Cursor position to access the probe tool;
3. Use the probe to identify the coordinates of points of the *plot* between , as in $[[x_1, y_1], [x_2, y_2], \ldots, [x_n, y_n]]$.
4. Search the *help* system for '*interpolate*' and choose a command to interpolate a polynomial approximating $\sin(x)$ between $-$ pi and pi . How many points do you need to obtain an approximation more or less acceptable?

▶ *Solution*

---

3. Use *plots:-plotcompare* to determine for which values of z you have $\frac{1}{\sqrt{z}} \neq \sqrt{\frac{1}{z}}$.

Try it with the options *same_box*, *assuming* z::real and *assuming* z::imaginary

▶ *Solution*

▼ *2. Algebraic Expressions, Equations and Functions*

Algebraic expression	any mathematical object built with numbers, symbols and functions combined using arithmetic operations
Equation	a construction using the = sign, typically with algebraic expressions on the left-hand and right-hand sides
Function	It can be a *known* function (of type 'known') as $\ln(z)$ or $J_n(z)$, or an *unknown* function (of type 'known') for example $f(x, y, z, t)$
Mapping	maps variables into constructions that involve these variables, typically algebraic expressions, for example $f := (x, y, z) \mapsto \sqrt{x^2 + y^2 + z^2}$
Manipulation commands	• To represent function application use (), as in $f(x)$ • To construct expressions, equations and mappings use: =, :=, ->, and unapply to convert an expression into a mapping, • Related to expressions: numer, denom, collect, coeff, degree • Related to equations and inequations: =, <>, <=, >= and to get each side use lhs, rhs, • Basic manipulation of expressions and equations: subs, eval, map, collect, isolate, solve

Table 2: Algebraic expressions, equations and functions

▼ *Examples*

An algebraic expression

> *restart*;

> $a\,x^2 + \frac{e^x}{b}$

$$a\,x^2 + \frac{e^x}{b} \qquad \textbf{(3.2.2.1.1)}$$

Note you can think of the labels as names assigned any visible output.

You can also give any name to an expression (assign a name to it) in order to refer to it, and also if the expression is not displayed. You do that by using the assign operator :=

> $f :=$ **(3.2.2.1.1)**

$$f := a\,x^2 + \frac{e^x}{b} \qquad \textbf{(3.2.2.1.2)}$$

Now you can refer to the expression **(3.2.2.1.1)** using the given name

> f

$$a\,x^2 + \frac{e^x}{b} \qquad \textbf{(3.2.2.1.3)}$$

Different from an expression, an equation always has left and right-hand sides with the \`=\` operator in between. For example

> $f = 0$

$$a\,x^2 + \frac{e^x}{b} = 0 \qquad \textbf{(3.2.2.1.4)}$$

You get each of the sides using the lhs and rhs commands

> *lhs*(**(3.2.2.1.4)**)

$$a\,x^2 + \frac{e^x}{b} \qquad \textbf{(3.2.2.1.5)}$$

> *rhs*(**(3.2.2.1.4)**)

$$0 \qquad \textbf{(3.2.2.1.6)}$$

You can assign names to everything, also to an equation

> $h := f = g$;

$$h := a\,x^2 + \frac{e^x}{b} = g \qquad \textbf{(3.2.2.1.7)}$$

What we call "the function of x equal to $a\,x^2 + \frac{e^x}{b}$" is implemented in the computer as a *mapping,* using the arrow operator ->

> $x \rightarrow$ **(3.2.2.1.1)**

$$x \mapsto a\,x^2 + \frac{e^x}{b} \qquad \textbf{(3.2.2.1.8)}$$

To use a mappings it is also practical to assign a name to it

> $h := x \rightarrow$ **(3.2.2.1.1)**

$$h := x \mapsto a\,x^2 + \frac{e^x}{b} \qquad \textbf{(3.2.2.1.9)}$$

> $h(x)$

$$a\,x^2 + \frac{e^x}{b} \qquad \textbf{(3.2.2.1.10)}$$

> **(3.2.2.1.8)**(x)

$$a\,x^2 + \frac{e^x}{b} \qquad \textbf{(3.2.2.1.11)}$$

Note however that the *mapping* h not really a function of x, but also of whatever argument you pass to it, as in

> $h(y)$

$$a\,y^2 + \frac{e^y}{b} \qquad \textbf{(3.2.2.1.12)}$$

> $h(\text{alpha})$

$$a\,\alpha^2 + \frac{e^\alpha}{b} \qquad \textbf{(3.2.2.1.13)}$$

> **(3.2.2.1.8)**(beta)

$$a\,\beta^2 + \frac{e^\beta}{b} \qquad \textbf{(3.2.2.1.14)}$$

You can convert an expression or equation into a mapping using unapply

> **(3.2.2.1.1)**

$$a\,x^2 + \frac{e^x}{b} \qquad \textbf{(3.2.2.1.15)}$$

> $\mathit{unapply}(\%, x)$

$$x \mapsto a\,x^2 + \frac{e^x}{b} \qquad \textbf{(3.2.2.1.16)}$$

> $\mathit{unapply}($**(3.2.2.1.1)**$, x, a, b)$

$$(x, a, b) \mapsto a\,x^2 + \frac{e^x}{b} \qquad \textbf{(3.2.2.1.17)}$$

Returning to the expression f

> f

$$a\,x^2 + \frac{e^x}{b} \qquad \textbf{(3.2.2.1.18)}$$

You can get the numerator, denominator or the coefficient of a or of $b^{(-1)}$, or compute the maximum and minimum degrees with respect to any variable

> $\mathit{numer}(f)$

$$a\,x^2\,b + e^x \qquad \textbf{(3.2.2.1.19)}$$

> $\mathit{denom}(f)$

$$b \qquad \textbf{(3.2.2.1.20)}$$

> $coeff(f, a)$

$$x^2 \quad (3.2.2.1.21)$$

> $coeff(f, b, -1)$

$$e^x \quad (3.2.2.1.22)$$

> $degree(f, b), ldegree(f, b)$

$$0, -1 \quad (3.2.2.1.23)$$

> $degree(f, x)$

$$FAIL \quad (3.2.2.1.24)$$

> $degree(f, e^x)$

$$1 \quad (3.2.2.1.25)$$

> $frontend(degree, [f, x])$

$$2 \quad (3.2.2.1.26)$$

You can substitute into or solve expressions and equations

> $subs(x = 0, f)$

$$\frac{e^0}{b} \quad (3.2.2.1.27)$$

Note the difference with eval: it *evaluates* the function

> $eval(f, x = 0)$

$$\frac{1}{b} \quad (3.2.2.1.28)$$

Most functions automatically return a value for their simplest special cases, as e^0. Inert functions are useful to avoid these automatic simplifications, for example:

> $[\%exp(0) = \exp(0), \%sin(0) = \sin(0), \%cos(0) = \cos(0)]$

$$[e^0 = 1, \sin(0) = 0, \cos(0) = 1] \quad (3.2.2.1.29)$$

You can activate inert functions using the value command

> $value(\textbf{(3.2.2.1.29)})$

$$[1 = 1, 0 = 0, 1 = 1] \quad (3.2.2.1.30)$$

The mathematical properties of the inert functions are understood by the system

> $\%sin(\%cos(z))$

$$\sin(\cos(z)) \quad (3.2.2.1.31)$$

> $diff(\textbf{(3.2.2.1.31)}, z)$

$$-\sin(z)\cos(\cos(z)) \quad (3.2.2.1.32)$$

You can solve expressions or equations or systems of them. When solving, an expression is considered an equation with right-hand side equal to zero

> f

$$a x^2 + \frac{e^x}{b} \quad (3.2.2.1.33)$$

> $isolate(f, b)$

(3.2.2.1.34)

$$b = -\frac{e^x}{a\,x^2} \qquad \textbf{(3.2.2.1.34)}$$

> *solve*(*f*, {*b*})

$$\left\{b = -\frac{e^x}{a\,x^2}\right\} \qquad \textbf{(3.2.2.1.35)}$$

isolate however only returns one solution. To get all the solutions use solve

> *isolate*(*f*, *x*)

$$x = -2\,W\left(-\frac{\sqrt{-\frac{1}{a\,b}}}{2}\right) \qquad \textbf{(3.2.2.1.36)}$$

> *solve*(*f*, {*x*})

$$\left\{x = -2\,W\left(-\frac{\sqrt{-\frac{1}{a\,b}}}{2}\right)\right\}, \left\{x = -2\,W\left(\frac{\sqrt{-\frac{1}{a\,b}}}{2}\right)\right\} \qquad \textbf{(3.2.2.1.37)}$$

> *lprint*(**(3.2.2.1.37)**)

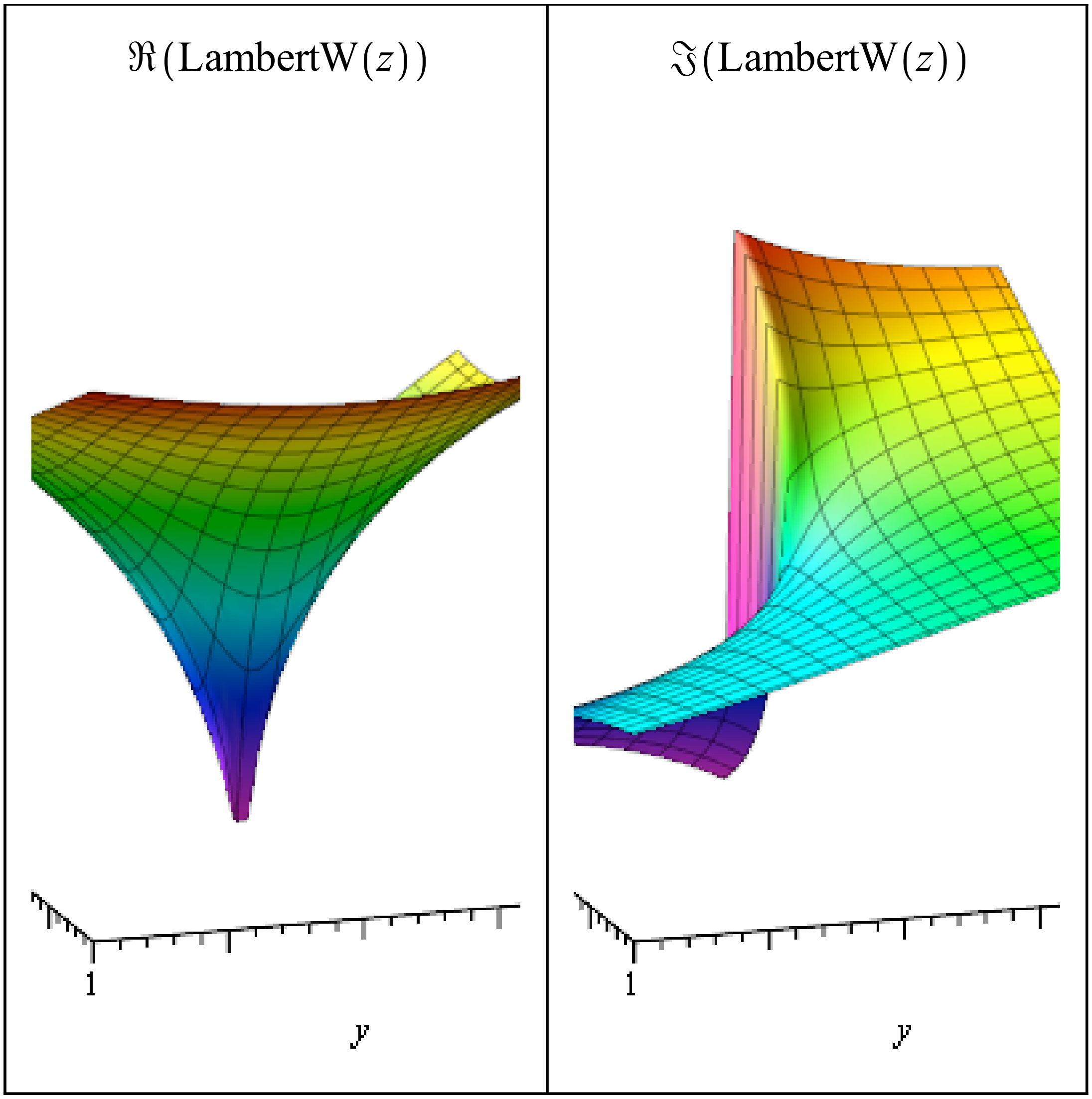
```
{x = -2*LambertW(-(1/2)*(-1/(a*b))^(1/2))}, {x = -2*
LambertW((1/2)*(-1/(a*b))^(1/2))}
```

> *solve*(*f*, {*x*}, *AllSolutions*)

$$\left\{x = -2\,W\left(\_Z2, -\frac{\sqrt{-\frac{1}{a\,b}}}{2}\right)\right\}, \left\{x = -2\,W\left(\_Z3, \frac{\sqrt{-\frac{1}{a\,b}}}{2}\right)\right\} \qquad \textbf{(3.2.2.1.38)}$$

> *PlotExpression* := *f* → *plots:-plotcompare*(*f*, 0, *_rest*, '*expression_plot*', 5) :

> *PlotExpression*(LambertW(*z*))

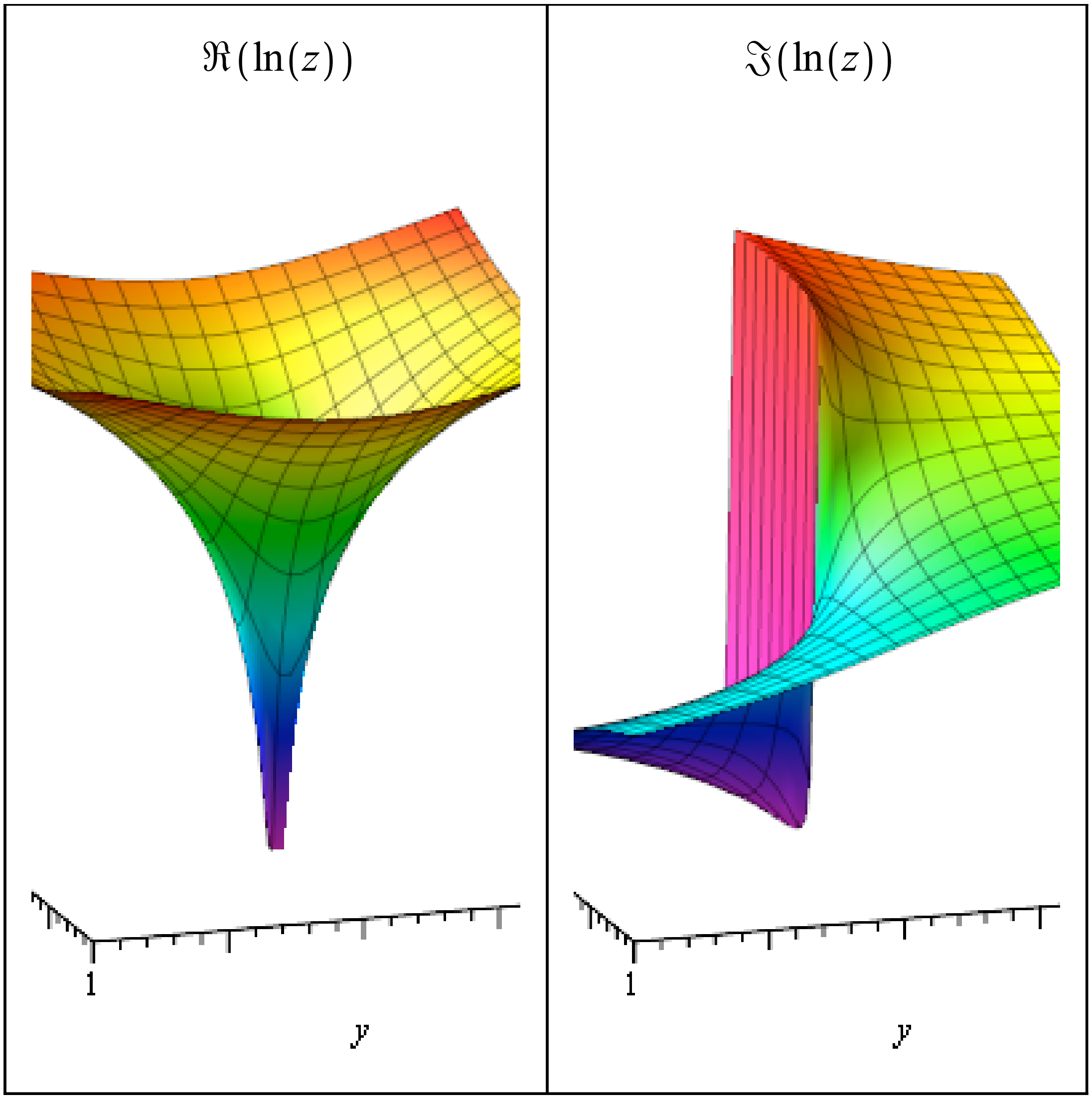


> *FunctionAdvisor*(*cuts*, LambertW)

```
* Partial match of "cuts" against topic "branch_cuts".
```

$$\left[W(z), z < -\frac{1}{e}\right], \left[W(a, z), (a \neq 0 \wedge z < 0) \vee \left(a = 0 \wedge z < -\frac{1}{e}\right)\right] \quad \textbf{(3.2.2.1.39)}$$

> *PlotExpression*(ln(z))



> *FunctionAdvisor*(*def*, LambertW(*z*))

```
* Partial match of "def" against topic "definition".
```

$$\left[W(z) = 1 + e^{\frac{\frac{I}{2}\left(\int_0^{\infty} \frac{\ln\left(\frac{\_k1 - I\pi - \ln(\_k1) + \ln(z)}{\_k1 + I\pi - \ln(\_k1) + \ln(z)}\right)}{\_k1 + 1} \, d\_k1\right)}{\pi}} (-1 + \ln(z)), \neg z :: \left[-\frac{1}{e}, 0\right] \right] \quad \textbf{(3.2.2.1.40)}$$

>

Exercises

1. Consider what we call $f(x) = \cos(x)^2 + g(x)$

a. Enter the expression $\cos(x)^2 + g(x)$

b. Use % to refer to this expression and assign the name F to it

c. Compute the value of F for: $x = \mathrm{pi}$ and $x = I\,a$

d. Transform F into a mapping of x and assign the name G to it

e. Use the mapping G to compute the values of item c

▶ *Solution*

2. Construct a polynomial of 2nd degree taking the product of monomials of the form ($x - \mathrm{alpha}_j$) where alpha_j are the roots and compute the maximum and minimum degrees with respect to x, then the coefficients of x to the powers 2,1 and 0, one at a time (coeff) or all at once (coeffs)

▶ *Solution*

3. Consider the transformation equations between cartesian and spherical coordinates

$$x = r\sin(\mathrm{theta})\cos(\mathrm{phi}), \quad y = r\sin(\mathrm{theta})\sin(\mathrm{phi}), \quad z = r\cos(\mathrm{theta})$$

Use the commands *isolate*, *map* and *subs* - and *assuming* to tell the domain of r, theta and phi - in order to invert these equations

▶ *Solution*

▼ *3. Limits, Derivatives, Sums, Products, Integrals, Differential Equations*

Commands	limit, diff and D, sum, product, int, dsolve, pdsolve
Manipulation commands	PDEtools:-dchange, PDEtools:-casesplit, the inert forms %limit, %int, etc. and the related value command

Table 3: Calculus

▼ *Examples*

The commands to compute limits, derivatives, sums and products are limit, diff, sum, product. The D command also represents derivatives - more about this afterwards.

> *restart*;

> $'limit'\left(\frac{\sin(x)}{x}, x = 0\right)$

$$\lim_{x \to 0} \frac{\sin(x)}{x} \quad \textbf{(3.2.3.1.1)}$$

> **(3.2.3.1.1)**

$$1 \quad \textbf{(3.2.3.1.2)}$$

All Maple commands have an inert version of them, that represent the mathematical object but does not perform the computation until you require it using the value command. Inert subexpressions always have some part displayed in grey:

> $\%limit\left(\frac{\sin(x)}{x}, x = 0\right)$

$$\lim_{x \to 0} \frac{\sin(x)}{x} \qquad \textbf{(3.2.3.1.3)}$$

> $value(\textbf{(3.2.3.1.3)})$

$$1 \qquad \textbf{(3.2.3.1.4)}$$

> $subs\left(x = t, \%diff\left(g(x) + \exp\left(x^2\right), x\right)\right)$

$$\frac{d}{dt}\left(g(t) + e^{t^2}\right) \qquad \textbf{(3.2.3.1.5)}$$

> $value(\textbf{(3.2.3.1.5)})$

$$\frac{d}{dt} g(t) + 2\, t\, e^{t^2} \qquad \textbf{(3.2.3.1.6)}$$

Handy: functionality is distributed over the sides of equations, so you can write this:

$$(\%sum = sum)\left(\frac{x^n}{n!}, n = 0 ..infinity\right)$$

directly as

> $(\%sum = sum)\left(\frac{x^n}{n!}, n = 0 ..infinity\right)$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \qquad \textbf{(3.2.3.1.7)}$$

All the family of *sum*, *int*, *solve*, *dsolve* and *pdsolve* are rather powerful commands nowadays. In the case of summation, note that it can also be performed in the indefinite case with the meaning:

$$\sum_k f(k) = g(k) \quad \text{where} \quad g(k+1) - g(k) = f(k)$$

When entering the following command, you will be asked whether it represents a function definition or a remember table assignment, choose remember table assignment (to perform these assignments with a function on the left-hand side without being asked questions enter first you can also enter $Typesetting\text{:-}Settings('functionassign = false')$)

> $f(k) := \left(\frac{k}{2}\right)!\, k$

$$f := k \mapsto \left(\frac{k}{2}\right)!\, k \qquad \textbf{(3.2.3.1.8)}$$

> $\%sum(f(k), k);$

$$\sum_k \left(\frac{k}{2}\right)!\, k \qquad \textbf{(3.2.3.1.9)}$$

> *value*((**3.2.3.1.9**))

$$2\left(\frac{k}{2}\right)! + 2\left(\frac{k}{2}+\frac{1}{2}\right)! \qquad \textbf{(3.2.3.1.10)}$$

> *eval*((**3.2.3.1.10**), $k = \mathrm{k} + 1$) - (**3.2.3.1.10**)

$$2\left(\frac{k}{2}+1\right)! - 2\left(\frac{k}{2}\right)! \qquad \textbf{(3.2.3.1.11)}$$

> *simplify*(%)

$$\left(\frac{k}{2}\right)!\, k \qquad \textbf{(3.2.3.1.12)}$$

Results are frequently expressed in terms of not-so-familiar special functions

> $\%int\left(\mathrm{e}^{-x^2}, x\right)$;

$$\int \mathrm{e}^{-x^2}\,\mathrm{d}x \qquad \textbf{(3.2.3.1.13)}$$

> *value*((**3.2.3.1.13**))

$$\frac{\sqrt{\pi}\,\operatorname{erf}(x)}{2} \qquad \textbf{(3.2.3.1.14)}$$

> $\%int\left(\frac{1}{\operatorname{sqrt}(2\,t^4 - 3\,t^2 - 2)}, t = 2..3\right)$

$$\int_2^3 \frac{1}{\sqrt{2\,t^4 - 3\,t^2 - 2}}\,\mathrm{d}t \qquad \textbf{(3.2.3.1.15)}$$

> *value*((**3.2.3.1.15**))

$$\frac{\sqrt{5}\,F\left(\frac{\sqrt{7}}{3}, \frac{\sqrt{5}}{5}\right)}{5} - \frac{\sqrt{5}\,F\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{5}}{5}\right)}{5} \qquad \textbf{(3.2.3.1.16)}$$

> *lprint*(%)

```
(1/5)*5^(1/2)*EllipticF((1/3)*7^(1/2), (1/5)*5^(1/2))-
(1/5)*5^(1/2)*EllipticF((1/2)*2^(1/2), (1/5)*5^(1/2))
```

Most of these commands have options to workaround special cases

> $int\left(\frac{1}{x}, x = a..2\right)$

```
Warning, unable to determine if 0 is between a and 2; 
try to use assumptions or use the AllSolutions option
```

$$\int_a^2 \frac{1}{x}\,\mathrm{d}x \qquad \textbf{(3.2.3.1.17)}$$

> $int\left(\frac{1}{x}, x = a..2, \text{'AllSolutions'}\right)$

$$\begin{cases} \mathit{undefined} & a < 0 \\ \infty & a = 0 \\ -\ln(a) + \ln(2) & 0 < a \end{cases} \qquad \textbf{(3.2.3.1.18)}$$

The *assuming* command is also handy in these cases

> **(3.2.3.1.17)** assuming $a > 0$;

$$-\ln(a) + \ln(2) \quad \textbf{(3.2.3.1.19)}$$

> **(3.2.3.1.17)** assuming $a < 0$;

$$\mathit{undefined} \quad \textbf{(3.2.3.1.20)}$$

The ordinary and partial differential equation commands have by now concentrated so much solving power that themselves are used to develop new solving algorithms

> *restart*;

> $\mathit{infolevel}[\mathit{dsolve}] := 5$

$$\mathit{infolevel}_{\mathit{dsolve}} := 5 \quad \textbf{(3.2.3.1.21)}$$

> $\mathit{PDEtools}{:-}\mathit{declare}(y(x), \mathit{prime} = x)$

y(x) will now be displayed as y

derivatives with respect to x of functions of one variable will now be displayed with ' **(3.2.3.1.22)**

> $\mathit{ode}_2 := y'(x) - y(x)^2 + (y(x)\sin(x)) - \cos(x) = 0$

$$\mathit{ode}_2 := y' - y^2 + y\sin(x) - \cos(x) = 0 \quad \textbf{(3.2.3.1.23)}$$

> $\mathit{dsolve}(\mathit{ode}[2])$

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
Chini's absolute invariant is: (1/4)*sin(x)^2*(2*cos
(x)-1)^2/cos(x)^3
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati with symmetry of the form [0, exp(-Int
(f,x))/P*(y*P-f)^2] successful
```

$$y = -\frac{e^{-\cos(x)}}{\_C1 + \int e^{-\cos(x)}\,dx} + \sin(x) \quad \textbf{(3.2.3.1.24)}$$

> $\mathit{ode}_3 := y'(x) = \dfrac{x\left(-x - 1 + x^2 - 2x^2 y(x) + (2x^4)\right)}{\left((x^2 - y(x))(x+1)\right)}$

$$\mathit{ode}_3 := y' = \frac{x\left(x^2 - x - 1 - 2x^2 y + 2x^4\right)}{(x^2 - y)(x+1)} \quad \textbf{(3.2.3.1.25)}$$

Computing an integrating factor

> $\mathit{DEtools}[\mathit{intfactor}](\mathit{ode}[3])$

```
 -> Computing symmetries using: way = 2
```

$$\left[0, \frac{-\frac{1}{2} - \frac{1}{2}x + y - x^2 - x^3 + xy}{(x^2 - y)(x+1)}\right]$$

```
<- successful computation of symmetries.
```

$$\frac{-x^2 + y}{-2x^2 + 2y - 1} \qquad (3.2.3.1.26)$$

The product of an ode and its integrating factor results in an total derivative

> **(3.2.3.1.26) (3.2.3.1.25)**

$$\frac{(-x^2 + y)\, y'}{-2x^2 + 2y - 1} = \frac{(-x^2 + y)\, x\, (x^2 - x - 1 - 2x^2 y + 2x^4)}{(-2x^2 + 2y - 1)(x^2 - y)(x+1)} \qquad (3.2.3.1.27)$$

From where

> *dsolve*(**(3.2.3.1.27)**)

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful
```

$$y = -\frac{1}{2(x+1)^4}\left(-2x^6 - 8x^5 - 13x^4 - 12x^3 - 8x^2 + e^{-W\left(-\frac{e^{\frac{4x^3}{3}} (e^x)^4 e^{-1}}{(e^{x^2})^4 (e^{\_C1})^4 (x+1)^4}\right) + \frac{4x^3}{3} - 4x^2 - 4\_C1 + 4x - 1} - 4x - 1\right) \qquad (3.2.3.1.28)$$

> *simplify*(**(3.2.3.1.28)**)

$$y = x^2 + \frac{W\left(-\frac{e^{\frac{4}{3}x^3 - 4x^2 - 4\_C1 + 4x - 1}}{(x+1)^4}\right)}{2} + \frac{1}{2} \qquad (3.2.3.1.29)$$

The notation for special functions is frequently unfamiliar - use lprint

> *lprint*(**(3.2.3.1.29)**)

```
y(x) = x^2+(1/2)*LambertW(-exp((4/3)*x^3-4*x^2-4*_C1+4*
x-1)/(x+1)^4)+1/2
```

Laplace equation in spherical coordinates:

> *restart*;

> *PDEtools*:-*declare*(*F*(*r*, theta, phi))

(3.2.3.1.30)

$$F(r,\theta,\phi)\ \textit{will now be displayed as}\ F \tag{3.2.3.1.30}$$

> $PDE := \%diff(r\verb|^|2 * \%diff(F(r, \text{theta}, \text{phi}), r), r)$
> $+ 1/\sin(\text{theta}) * \%diff(\sin(\text{theta}) * \%diff(F(r, \text{theta}, \text{phi}), \text{theta}), \text{theta})$
> $+ 1/\sin(\text{theta})\verb|^|2 * \%diff(F(r, \text{theta}, \text{phi}), \text{phi}, \text{phi}) = 0$

$$PDE := \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} F\right) + \frac{\frac{\partial}{\partial \theta}\left(\sin(\theta) \frac{\partial}{\partial \theta} F\right)}{\sin(\theta)} + \frac{\frac{\partial^2}{\partial \phi^2} F}{\sin(\theta)^2} = 0 \tag{3.2.3.1.31}$$

> $PDE := value(PDE)$

$$PDE := 2\, r\, F_r + r^2 F_{r,r} + \frac{\cos(\theta)\, F_\theta + \sin(\theta)\, F_{\theta,\theta}}{\sin(\theta)} + \frac{F_{\phi,\phi}}{\sin(\theta)^2} = 0 \tag{3.2.3.1.32}$$

The standard solution separating variables by product

> $Physics{:-}Setup(auto = true)$

** Partial match of 'auto' against keyword 'automaticsimplification'*

$$[automaticsimplification = true] \tag{3.2.3.1.33}$$

> $pdsolve(PDE, build)$

$$F = \frac{1}{\sqrt{r}\ \mathrm{csgn}(\sin(\theta))}\left(\sqrt{2}\left(-\sin(\theta)^2\right)^{\frac{\sqrt{\_c_2}}{2}}\left(\_C5 \sin\left(\sqrt{\_c_2}\ \phi\right) + \_C6 \cos\left(\sqrt{\_c_2}\ \phi\right)\right)\left(\_C1\, r^{\frac{\sqrt{1+4\_c_1}}{2}} + \_C2\, r^{-\frac{\sqrt{1+4\_c_1}}{2}}\right)\left(\cos(\theta)\ {}_2F_1\left(\frac{\sqrt{\_c_2}}{2} + \frac{\sqrt{1+4\_c_1}}{4} + \frac{3}{4}, \frac{\sqrt{\_c_2}}{2} - \frac{\sqrt{1+4\_c_1}}{4} + \frac{3}{4}; \frac{3}{2}; \frac{\cos(2\theta)}{2} + \frac{1}{2}\right)\_C3 + \mathrm{csgn}(\sin(\theta))\ \_C4\ {}_2F_1\left(\frac{\sqrt{\_c_2}}{2} + \frac{\sqrt{1+4\_c_1}}{4} + \frac{1}{4}, \frac{\sqrt{\_c_2}}{2} - \frac{\sqrt{1+4\_c_1}}{4} + \frac{1}{4}; \frac{1}{2}; \frac{\cos(2\theta)}{2} + \frac{1}{2}\right)\right)\right) \tag{3.2.3.1.34}$$

Laplace equation also admits a solution separable by sum

> $pdsolve(PDE, HINT = `+`)$

$$(F = \_F1(r) + \_F2(\theta) + \_F3(\phi))\ \&\text{where}\left[\left\{\_F1_{r,r} = \frac{-2\,\_F1_r\, r + \_c_1}{r^2}, \right.\right. \tag{3.2.3.1.35}$$

$$\_F2_{\theta,\theta} = -\_c_1 - \frac{\cos(\theta)\ \_F2_{\theta}}{\sin(\theta)} - \frac{\_c_3}{\sin(\theta)^2}, \_F3_{\phi,\phi} = \_c_3\}\Big]\Big]$$

You transform these structures into a concrete solution using the build command, or using the build option as in pdsolve(PDE, build). For example

> $PDEtools\text{:-}build(\%)$

$$F = \frac{1}{2r}\left(-\_c_3 \ln\left(\frac{I - I e^{I\theta}}{e^{I\theta}+1}\right)^2 r + 2r\left(\_c_1 + \_C3\right)\ln\left(\frac{I - I e^{I\theta}}{e^{I\theta}+1}\right) - r\left(\_c_1 - \_C3\right)\ln\left(-\frac{e^{I\theta}}{\left(e^{I\theta}+1\right)^2}\right) - r\left(\_c_1 + \_C3\right)\ln\left(\frac{e^{I\theta}}{\left(e^{I\theta}+1\right)^2}\right) - 4\_c_1\ln(2)\, r + 2\_c_1\ln(r)\, r + \left(\_c_3\phi^2 + 2\_C5\,\phi + 2\_C2 + 2\_C4 + 2\_C6\right) r - 2\_C1\right) \qquad \textbf{(3.2.3.1.36)}$$

Symmetry methods and systems of partial differential equations, linear and nonlinear, can be solved in many cases.

A nonlinear ODE and the linear PDE system for its symmetries

> $PDEtools\text{:-}declare(\,(\mathrm{xi}, \mathrm{eta})(x,y), y(x), prime = x)$

ξ(x, y) will now be displayed as ξ

η(x, y) will now be displayed as η

y(x) will now be displayed as y

derivatives with respect to x of functions of one variable will now be displayed with ' **(3.2.3.1.37)**

> $ode_{11} := y''(x) + \left(a\,x^r\,y(x)^n\right) = 0$

$$ode_{11} := y'' + a\,x^r\,y^n = 0 \qquad \textbf{(3.2.3.1.38)}$$

> $sys := map(u \to u = 0, [DEtools[gensys](\ ode[11], [\mathrm{xi}, \mathrm{eta}](x,y)\ )]);$

$$sys := \left[\xi_{y,y} = 0, \eta_{y,y} - 2\,\xi_{x,y} = 0, 3\,y^n\,\xi_y\,x^r\,a + 2\,\eta_{x,y} - \xi_{x,x} = 0, \frac{\eta_{x,x}\,y\,x + 2\,x^r\left(\xi_x\,x\,y - \frac{\eta_y\,x\,y}{2} + \frac{n\,x\,\eta}{2} + \frac{r\,y\,\xi}{2}\right) a\,y^n}{y\,x} = 0\right] \qquad \textbf{(3.2.3.1.39)}$$

> $PDEtools\text{:-}casesplit(sys)$

$$\left[\eta = -\frac{\xi\,y\,(r+2)}{x\,(n-1)}, \xi_x = \frac{\xi}{x}, \xi_y = 0\right] \ \&\text{where}\ [\ ] \qquad \textbf{(3.2.3.1.40)}$$

> $pdsolve(sys)$

$$\left\{\eta = -\frac{\_C1\, y\,(r+2)}{n-1},\ \xi = \_C1\, x\right\} \tag{3.2.3.1.41}$$

A solution for this system such that r is a parameter (so, also a solving variable) and n is different from 1

> $sys_2 := [DEtools[gensys](\, ode[11],\, [\mathrm{xi}, \mathrm{eta}](x, y)\,),\, n \neq 1];$

$$sys_2 := \left[\xi_{y,y},\ \eta_{y,y} - 2\,\xi_{x,y},\ 3\,y^n\,\xi_y\,x^r\,a + 2\,\eta_{x,y} - \xi_{x,x},\ \frac{\eta_{x,x}\,y\,x + 2\,x^r\left(\xi_x\,x\,y - \frac{\eta_y\,x\,y}{2} + \frac{n\,x\,\eta}{2} + \frac{r\,y\,\xi}{2}\right)a\,y^n}{y\,x},\ n \neq 1\right] \tag{3.2.3.1.42}$$

> $pdsolve\left(sys_2,\, [\mathrm{xi}, \mathrm{eta}, r]\right)$

$$\{r = -2, \eta = 0, \xi = \_C1\, x\},\ \{r = -n-3, \eta = 0, \xi = 0\},\ \left\{r = -n-3,\ \eta = (\_C1\, x + \_C2)\, y,\ \xi = \frac{x\,(n\,(\_C1\, x + \_C2) + \_C1\, x - \_C2)}{n+1}\right\},\ \left\{r = r,\ \eta = \_C1\, y,\ \xi = -\frac{\_C1\, x\,(n-1)}{r+2}\right\} \tag{3.2.3.1.43}$$

Indeed if you take r as a solving variable, using differential algebra techniques the problem splits into three different problems (so called: the general and the singular cases:

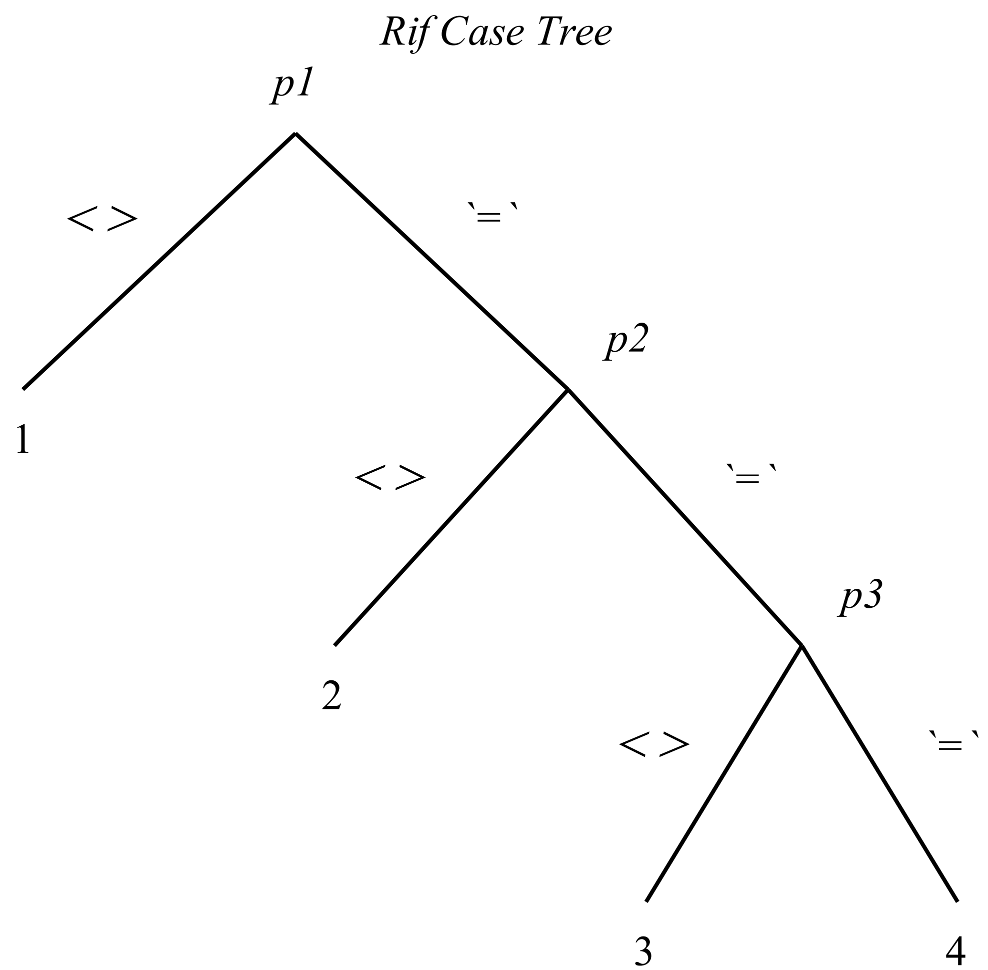
> $PDEtools\text{:-}casesplit\left(sys_2,\, [\mathrm{xi}, \mathrm{eta}, r],\, caseplot\right)$

========= *Pivots Legend* =========

$$p1 = (r+2)\,(r+n+3)$$

$$p2 = \eta$$

$$p3 = r + 2$$



$$\left[\xi = -\frac{\eta\, x\, (n-1)}{y\, (r+2)}, \eta_x = 0, \eta_y = \frac{\eta}{y}\right] \&\text{where} \left[(r+2)\, x^r \neq 0, (r+n+3)\, x^r \neq 0\right], \left[\xi = \frac{x\left(2\,\eta_x\, x + \eta\,(n-1)\right)}{y\,(n+1)}, \eta_{x,x} = 0, \eta_y = \frac{\eta}{y}, r = -n-3\right] \&\text{where} [\eta \neq 0], [\xi = 0, \eta = 0, r = -n-3] \&\text{where} [\ ], \left[\xi_x = \frac{\xi}{x}, \xi_y = 0, \eta = 0, r = -2\right] \&\text{where} [\ ] \tag{3.2.3.1.44}$$

>

Exercises

There is no much to say about all these commands but for dsolve and pdsolve, the solvers for ordinary and partial differential equations, as well as the PDEtools:-casesplit command for triangularizing systems of equations. So the exercises for this section are about exploration.

1. Open the help page for dsolve/education ,

a. Transform the page into a worksheet (one of the icons on the toolbar);

b. Go to the menu View -> Collapse All Sections, and choose a section you want to explore. My recommendation according to how useful it could be in physics computations:

* If you are not familiar with symmetry methods, the corresponding section can give you a rapid glimpse on how to tackle ODEs by discovering their symmetries and using them to construct solutions

* The section on singular solutions may open your eyes to something you are probably not aware of regarding differential equations. Singular solutions are frequently the ones that are relevant in physics models. The relevant command here is PDEtools:-casesplit

* The section on using "this or that method" has no mathematical insights but is useful information regarding flexibility for computing different forms of general ODE solutions

---

2. The same with the help page for PDEtools:-InvariantSolutions

---

3. The product of an integrating factor an a differential equation is a *Total Derivative.* Use DEtools[redode] to construct a second order ODE family having an integrating factor mu = $F(x)$ -- an arbitrary function -- such that the reduced ODE has the same integrating factor.

▶ *Solution*

4. An ODE of order n^{th} admits n integrating factors. Use DEtools[redode] to construct the most general third order ODE admitting the
following three integrating factors:

$$\left[y(x)\, x,\, x^{-\frac{1}{4}+\frac{1}{4}\sqrt{29}}\, y(x),\, x^{-\frac{1}{4}-\frac{1}{4}\sqrt{29}}\, y(x)\right]$$

then use DEtools[mutest] to verify that the obtained ODE admits these 3 expressions as integrating factors

▶ *Solution*

▼ *4. Algebraic manipulation: simplify, factorize, expand*

Commands	simplify, factor, expand, combine, collect and convert

Table 4: Algebraic manipulation

▼ *Examples*

Simplification is not really a well defined operation, but one based on common sense, and the desired result sometimes depends on particularities of the problem.

Among the most typical simplifications there is the one that *makes use of functions identities*

> $\sin(x)^2 + \cos(x)^2$

$$\sin(x)^2 + \cos(x)^2 \quad \textbf{(3.2.4.1.1)}$$

> *simplify*(**(3.2.4.1.1)**)

$$1 \quad \textbf{(3.2.4.1.2)}$$

Another typical simplification is the *simplification in size*

> $\frac{e^{-\frac{1\,x^2}{4}}\,2^{\frac{1}{4}}\,x^{\frac{3}{2}}}{4} + \frac{1\,e^{\frac{1\,x^2}{4}}\,2^{\frac{3}{4}}\,\sqrt{x}\,\sqrt{\text{pi}}\,\text{F}(x)}{8} + \frac{1\,e^{\frac{1\,x^2}{4}}\,2^{\frac{3}{4}}\,x^{\frac{5}{2}}\,\sqrt{\text{pi}}\,\text{F}(x)}{8}$

$$\frac{\sqrt{x}\,2^{1/4}\left(\sqrt{2}\,(x^2+1)\,\sqrt{\pi}\,F(x)\,e^{\frac{x^2}{4}} + 2\,e^{-\frac{x^2}{4}}\,x\right)}{8} \quad \textbf{(3.2.4.1.3)}$$

> *simplify*(%, *size*)

$$\frac{\sqrt{x}\,2^{1/4}\left(\sqrt{2}\,(x^2+1)\,\sqrt{\pi}\,F(x)\,e^{\frac{x^2}{4}} + 2\,e^{-\frac{x^2}{4}}\,x\right)}{8} \quad \textbf{(3.2.4.1.4)}$$

In other cases, it all depends on what is preferred

> $6\,(x+4)\,(x-1)$

$$(6\,x + 24)\,(x - 1) \quad \textbf{(3.2.4.1.5)}$$

> $6\,x^2 + 18\,x - 24$

$$6\,x^2 + 18\,x - 24 \quad \textbf{(3.2.4.1.6)}$$

These two expressions are equal, and both are 'simplified', so simplify does nothing

> *simplify*(**(3.2.4.1.5)**)

$$6\,(x+4)\,(x-1) \quad \textbf{(3.2.4.1.7)}$$

> *simplify*(**(3.2.4.1.6)**)

$$6\,x^2 + 18\,x - 24 \quad \textbf{(3.2.4.1.8)}$$

To rewrite one as the other one, the operations to be performed are: to *factor* or to *expand*

> *factor*(**(3.2.4.1.5)**)

$$6\,(x+4)\,(x-1) \quad \textbf{(3.2.4.1.9)}$$

> *expand*(**(3.2.4.1.9)**)

$$6\,x^2 + 18\,x - 24 \quad \textbf{(3.2.4.1.10)}$$

In this case the expanded form is also a form where powers of x are collected, as in

> *collect*(**(3.2.4.1.9)**, x)

$$6\,x^2 + 18\,x - 24 \quad \textbf{(3.2.4.1.11)}$$

One of the most powerful simplifications is to simplify with respect to given equations, for example: "simplify $6\,x^2 + 18\,x - 24$ taking x+4=alpha and x-1=beta

> *simplify*(**(3.2.4.1.11)**, $\{x + 4 = \text{alpha}\}$)

$$6\,\alpha^2 - 30\,\alpha \quad \textbf{(3.2.4.1.12)}$$

Both expand and combine take into account the properties of mathematical functions, with the combine command rewriting powers of trigonometric functions as expressions linear in other trigonometric functions

> $\sin(x)^2 - \cos(x)^2$

$$\sin(x)^2 - \cos(x)^2 \quad \textbf{(3.2.4.1.13)}$$

> $combine(\textbf{(3.2.4.1.13)})$

$$-\cos(2x) \quad \textbf{(3.2.4.1.14)}$$

> $\%sum\left(a[j]\cdot\cos(j\cdot x)^j + b[j]\sin(j\,x)^{2j}, j = 1..2\right)$

$$\sum_{j=1}^{2}\left(a_j \cos(j\,x)^j + b_j \sin(j\,x)^{2j}\right) \quad \textbf{(3.2.4.1.15)}$$

> $value(\textbf{(3.2.4.1.15)})$

$$a_1 \cos(x) + b_1 \sin(x)^2 + a_2 \cos(2x)^2 + b_2 \sin(2x)^4 \quad \textbf{(3.2.4.1.16)}$$

> $combine(\textbf{(3.2.4.1.16)})$

$$\frac{(4a_2 - 4b_2)\cos(4x)}{8} + a_1\cos(x) - \frac{b_1\cos(2x)}{2} + \frac{b_2\cos(8x)}{8} + \frac{a_2}{2} + \frac{b_1}{2} + \frac{3b_2}{8} \quad \textbf{(3.2.4.1.17)}$$

We almost always want to 'simplify in size':

> $simplify(\textbf{(3.2.4.1.17)}, size)$

$$\frac{(4a_2 - 4b_2)\cos(4x)}{8} + a_1\cos(x) - \frac{b_1\cos(2x)}{2} + \frac{b_2\cos(8x)}{8} + \frac{a_2}{2} + \frac{b_1}{2} + \frac{3b_2}{8} \quad \textbf{(3.2.4.1.18)}$$

In this following example the simplification in size is more convenient that a direct simplification

> $e2 := -\left(3\sin(x)^{\frac{1}{2}}\cos(x)^2\sin(x)^m\right) + \left(3\sin(x)^{\frac{1}{2}}\cos(x)^2\cos(x)^n\right) + \left(4\sin(x)^{\frac{1}{2}}\cos(x)^4\sin(x)^m\right) - 4\sin(x)^{\frac{1}{2}}\cos(x)^4\cos(x)^n$

$$e2 := -4\sqrt{\sin(x)}\,\cos(x)^2\left(\cos(x)^2 - \frac{3}{4}\right)\left(\cos(x)^n - \sin(x)^m\right) \quad \textbf{(3.2.4.1.19)}$$

> $simplify(e2, size)$

$$-4\sqrt{\sin(x)}\,\cos(x)^2\left(\cos(x)^2 - \frac{3}{4}\right)\left(\cos(x)^n - \sin(x)^m\right) \quad \textbf{(3.2.4.1.20)}$$

> $simplify(e2)$

$$-4\sqrt{\sin(x)}\,\cos(x)^2\left(\cos(x)^2 - \frac{3}{4}\right)\left(\cos(x)^n - \sin(x)^m\right) \quad \textbf{(3.2.4.1.21)}$$

Other times what we really want is not a *simplification* but to have an expression rewritten with powers of same variables factored out (we say 'collected'), for example

> $p := b\,y + 6\,g\,x + x\,y$

$$p := (6g + y)\,x + b\,y \quad \textbf{(3.2.4.1.22)}$$

> $collect(p, [x, y])$

$$(6\,g + y)\,x + b\,y \tag{3.2.4.1.23}$$

> $collect(p, [y, x])$

$$(6\,g + y)\,x + b\,y \tag{3.2.4.1.24}$$

Sometimes all what we want is to cancel factors that appear in the numerator and denominator of an expressions, for example:

> $num := expand((x - a)\,(x - b))$

$$num := (b - x)\,(a - x) \tag{3.2.4.1.25}$$

> $den := expand((x - a)\,(x - c))$

$$den := (c - x)\,(a - x) \tag{3.2.4.1.26}$$

> $\frac{num}{den}$

$$\frac{b - x}{c - x} \tag{3.2.4.1.27}$$

To cancel common factors in the numerator and denominator we use *normal*

> $normal(\textbf{(3.2.4.1.27)})$

$$\frac{b - x}{c - x} \tag{3.2.4.1.28}$$

Finally what we sometimes want is just to rewrite an expression in terms of different functions, for example

> $\frac{1}{4}\,(e^x)^2 - \frac{\frac{1}{4}}{(e^x)^2}$

$$\frac{(e^x)^2}{4} - \frac{1}{4\,(e^x)^2} \tag{3.2.4.1.29}$$

> $simplify(\textbf{(3.2.4.1.29)})$

$$\frac{e^{2\,x}}{4} - \frac{e^{-2\,x}}{4} \tag{3.2.4.1.30}$$

> $convert(\textbf{(3.2.4.1.29)}, trig)$

$$\frac{(\cosh(x) + \sinh(x))^2}{4} - \frac{1}{4\,(\cosh(x) + \sinh(x))^2} \tag{3.2.4.1.31}$$

> $simplify(\textbf{(3.2.4.1.31)})$

$$\cosh(x)\,\sinh(x) \tag{3.2.4.1.32}$$

You can try converting any function into any other one, and the conversion will (almost always) proceed when the conversion is possible

>

Exercises

1. Show, algebraically, using *simplify* and *assuming*, that $\sqrt{z^2} = z$ when z is real and positive and discover the most general domain for z such that the identity holds

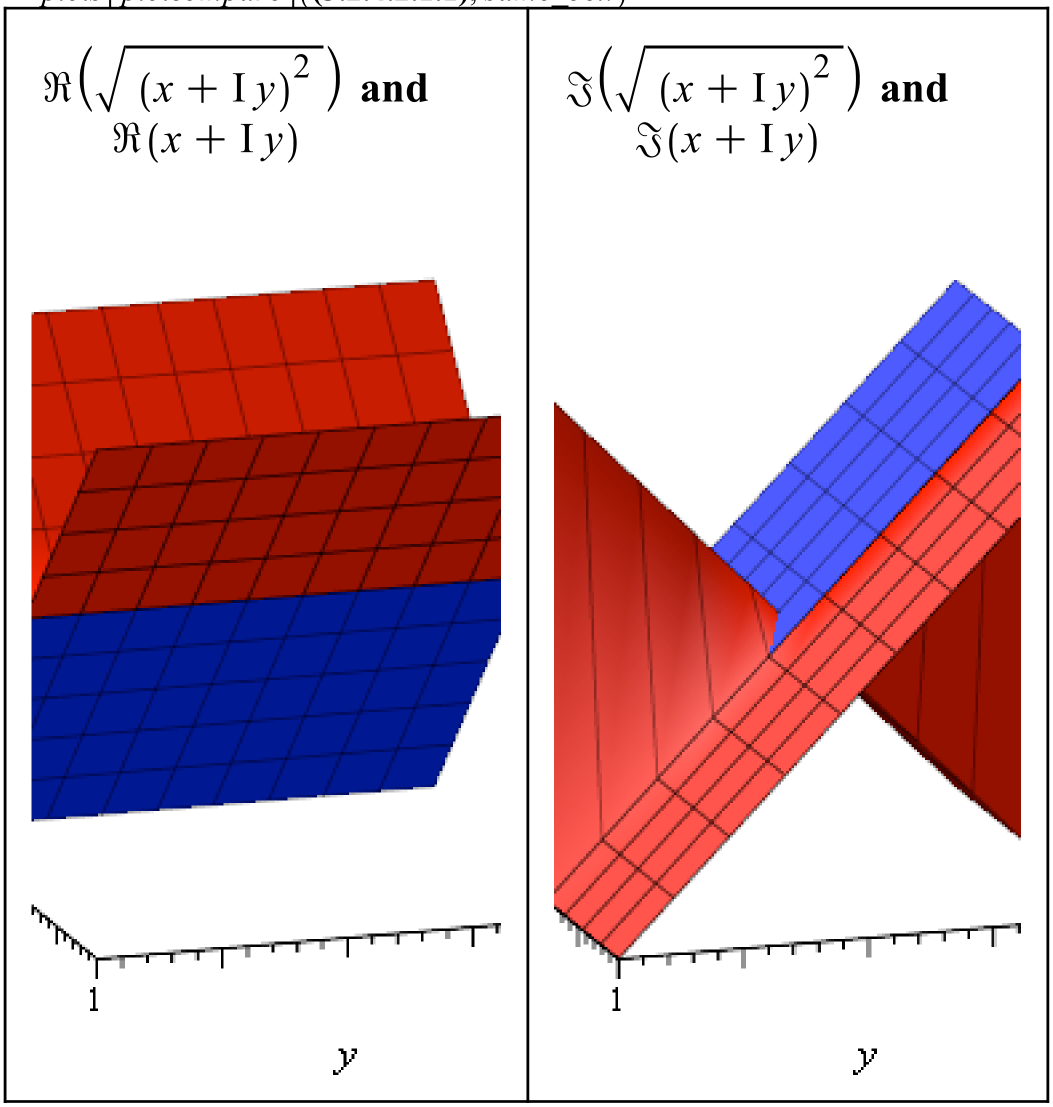
Solution

> sqrt$(z^2) = z$

$$\sqrt{z^2} = z \tag{3.2.4.2.1.1}$$

You can see this expression is not equal to z by comparing both expressions using plots[plotcompare]

> *plots*[*plotcompare*]((**3.2.4.2.1.1**), *same_box*)



Rotating the plots you see that these two functions are equal when $0 < z$. To simplify algebraically assuming that z > 0 , use

> *simplify*((**3.2.4.2.1.1**)) assuming $z > 0$;

$$z = z \tag{3.2.4.2.1.2}$$

To discover the most general domain for z such that the identity holds, by trial an error the first thing one could do is to try simplifying the expression as given:

> *simplify*((**3.2.4.2.1.1**))

$$\mathrm{csgn}(z)\, z = z \tag{3.2.4.2.1.3}$$

and, from its help page, csgn(z) is equal to 1 only when

$0 < \Re(z)$ **or** $(\Re(z) = 0$ **and** $0 < \Im(z))$.

So the following also simplifies to an identity

> *simplify*(**(3.2.4.2.1.1)**) assuming $\Re(z) = 0$ **and** $0 < \Im(z)$

$$\operatorname{csgn}(z)\, z = z \tag{3.2.4.2.1.4}$$

>

2. Use a *simplification taking into account that* $\sin^2 + \cos^2 = 1$ (see simplify,siderels) to show that

$$8 \sin(x)^4 \cos(x) + 15 \sin(x)^2 \cos(x)^3 - 15 \sin(x)^2 \cos(x) + 7 \cos(x)^5 - 14 \cos(x)^3 + 7 \cos(x)$$

is equal to 0.

Solution

> $f := 8 \sin(x)^4 \cos(x) + 15 \sin(x)^2 \cos(x)^3 - 15 \sin(x)^2 \cos(x) + 7 \cos(x)^5 - 14 \cos(x)^3 + 7 \cos(x)$

$$f := \cos(x) \left(8 \sin(x)^2 + 7 \cos(x)^2 - 7\right) \left(\cos(x)^2 + \sin(x)^2 - 1\right) \tag{3.2.4.2.2.1}$$

> $eq := \{\sin(x)^2 + \cos(x)^2 = 1\}$

$$eq := \{\sin(x)^2 + \cos(x)^2 = 1\} \tag{3.2.4.2.2.2}$$

> *simplify*(f, eq)

$$0 \tag{3.2.4.2.2.3}$$

>

5. *Matrices (Linear Algebra)*

Commands	Matrix, Vector is the same as Vector[colum], Vector[row], or matrix and vector. Use + and . for operations
Manipulation commands	LinearAlgebra package: conjugate, Transpose, HermitianTranspose, Determinant, Trace, Eigenvalues, Eigenvectors, MatrixExponential, LinearSolve linalg package: conjugate, transpose, htranspose, det, trace, eigenvalues, eigenvectors, exponential, linsolve

Table 5: Linear Algebra

Examples

There is a whole LinearAlgebra package with 130 commands to manipulate Matrices and solve linear algebra problems.
There is also the older linalg package with 114 matrix algebra commands.

Here we restrict to a small subset of matrix commands that are used more frequently, and for the rest: just consult help pages when necessary.

For historical and other reasons, there are two kind of matrices in Maple.

- The old ones, represented by the lowercase word matrix have the advantage that you can compute with them without displaying their contents.
- The new ones, represented by the word Matrix have the advantage of performing component computations faster

First *matrix*

> $A := matrix(2, 2, [a, b, c, d])$

$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{3.2.5.1.1}$$

Invoking the matrix does not show its components

> A

$$A \tag{3.2.5.1.2}$$

You can refer to an unspecified component (this is useful when setting brackets rules in Quantum Mechanics), as in

> $A[i, j]$

$$A_{i,j} \tag{3.2.5.1.3}$$

You any specified component by attributing values to the indices

> $eval(\mathbf{(3.2.5.1.3)}, [i = 1, j = 2])$

$$b \tag{3.2.5.1.4}$$

The same with *Matrix*

> $\mathrm{B} := \mathrm{Matrix}(2, 2, [a, b, c, d])$

$$B := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{3.2.5.1.5}$$

Invoking it shows its components

> B

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{3.2.5.1.6}$$

You cannot refer to an unspecified component

> $B[i, j]$

```
Error, bad index into Matrix
```

The *LinearAlgebra* package is all about *Matrix*, while there also exists the old *linalg* package about *matrix*. So you can do operations with both packages according to whether you need more symbolic capabilities (linalg) or faster computations (LinearAlgebra).

There are routines to convert a *matrix* into a *Matrix* and the other way around

> $C := convert(B, matrix)$

$$C := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \textbf{(3.2.5.1.7)}$$

> C

$$C \quad \textbf{(3.2.5.1.8)}$$

> $C[i,j]$

$$C_{i,j} \quad \textbf{(3.2.5.1.9)}$$

> $M := convert(A, Matrix)$

$$M := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \textbf{(3.2.5.1.10)}$$

> M

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \textbf{(3.2.5.1.11)}$$

> $M[i,j]$

Error, bad index into Matrix

Vectors can be represented using the *vector* and *Vector* commands

> $v := vector([v_a, v_b])$

$$v := \begin{bmatrix} v_a & v_b \end{bmatrix} \quad \textbf{(3.2.5.1.12)}$$

> v

$$v \quad \textbf{(3.2.5.1.13)}$$

> $v[j]$

$$v_j \quad \textbf{(3.2.5.1.14)}$$

> $v[2]$

$$v_b \quad \textbf{(3.2.5.1.15)}$$

> $V := Vector([V_a, V_b])$

$$V := \begin{bmatrix} V_a \\ V_b \end{bmatrix} \quad \textbf{(3.2.5.1.16)}$$

> V

$$\begin{bmatrix} V_a \\ V_b \end{bmatrix} \quad \textbf{(3.2.5.1.17)}$$

When using Matrix and Vector, summation and product are performed using `+` and `.`. When using matrix and vector, it is the same but you need to enclose the operation with *evalm*

> $v \,.\, A$

$$v \cdot A \quad \textbf{(3.2.5.1.18)}$$

> $evalm(\textbf{(3.2.5.1.18)})$

(3.2.5.1.19)

$$\left[\begin{array}{cc} v_a\,a+v_b\,c & v_a\,b+v_b\,d \end{array}\right] \tag{3.2.5.1.19}$$

Note that for Vector there are row and colum vectors, so

> $V.B$

Error, (in LinearAlgebra:-Multiply) cannot multiply a column Vector and a Matrix

> $V_{row} := Vector[row](V)$

$$V_{row} := \left[\begin{array}{cc} V_a & V_b \end{array}\right] \tag{3.2.5.1.20}$$

> $V_{row}\,.\,B$

$$\left[\begin{array}{cc} V_a\,a+V_b\,c & V_a\,b+V_b\,d \end{array}\right] \tag{3.2.5.1.21}$$

The typical operations: conjugate, Transpose, HermitianTranspose, Determinant, Trace, Eigenvalues

> $LinearAlgebra\text{:-}Determinant(B)$

$$a\,d-b\,c \tag{3.2.5.1.22}$$

> $LinearAlgebra\text{:-}Eigenvalues(B)$

$$\left[\begin{array}{c} \frac{d}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2\,a\,d+4\,b\,c+d^2}}{2} \\ \frac{d}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2\,a\,d+4\,b\,c+d^2}}{2} \end{array}\right] \tag{3.2.5.1.23}$$

> $conjugate(A)$

$$\overline{A} \tag{3.2.5.1.24}$$

> $evalm(\%)$

$$\left[\begin{array}{cc} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{array}\right] \tag{3.2.5.1.25}$$

> $LinearAlgebra\text{:-}Trace(B)$

$$d+a \tag{3.2.5.1.26}$$

Note these do not work with matrix for which you can use the old linalg

> $Trace(A)$

$$Trace(A) \tag{3.2.5.1.27}$$

> $linalg[trace](A)$

$$d+a \tag{3.2.5.1.28}$$

For solving linear systems there is the LinearSolve command.

Exercises

1. Determine the characteristic matrix, eigenvalues and then: step by step the eigenvectors, of the following matrix:

$$M = \begin{bmatrix} 0 & -\mathrm{I}\sqrt{2} & 0 \\ \mathrm{I}\sqrt{2} & 0 & -\mathrm{I}\sqrt{2} \\ 0 & \mathrm{I}\sqrt{2} & 0 \end{bmatrix}$$

Solution

> *restart*

> *with(LinearAlgebra)*

[&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, BilinearForm, CARE, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix, CompressedSparseForm, ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation, CrossProduct, DARE, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, FromCompressedSparseForm, FromSplitForm, GaussianElimination, GenerateEquations, GenerateMatrix, Generic, GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, KroneckerProduct, LA_Main, LUDecomposition, LeastSquares, LinearSolve, LyapunovSolve, Map, Map2, MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse, MatrixMatrixMultiply, MatrixNorm, MatrixPower, MatrixScalarMultiply, MatrixVectorMultiply, MinimalPolynomial, Minor, Modular, Multiply, NoUserValue, Norm, Normalize, NullSpace, OuterProductMatrix, Permanent, Pivot, PopovForm, ProjectionMatrix, QRDecomposition, RandomMatrix, RandomVector, Rank, RationalCanonicalForm, ReducedRowEchelonForm, Row, RowDimension, RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector, SchurForm, SingularValues, SmithForm, SplitForm, StronglyConnectedBlocks, SubMatrix, SubVector, SumBasis, SylvesterMatrix, SylvesterSolve, ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector, VandermondeMatrix, VectorAdd, VectorAngle, VectorMatrixMultiply, VectorNorm, VectorScalarMultiply, ZeroMatrix, ZeroVector, Zip] **(3.2.5.2.1.1)**

> $M := Matrix(3, (i,j) \rightarrow$ **if** abs$(i - j) = 1$ **then** $-I$ sqrt(2) **else** 0 **fi**, $shape = antisymmetric)$

$$M := \begin{bmatrix} 0 & -\mathrm{I}\sqrt{2} & 0 \\ \mathrm{I}\sqrt{2} & 0 & -\mathrm{I}\sqrt{2} \\ 0 & \mathrm{I}\sqrt{2} & 0 \end{bmatrix} \quad \textbf{(3.2.5.2.1.2)}$$

> $CharacteristicMatrix(M, x)$

$$\begin{bmatrix} x & \mathrm{I}\sqrt{2} & 0 \\ -\mathrm{I}\sqrt{2} & x & \mathrm{I}\sqrt{2} \\ 0 & -\mathrm{I}\sqrt{2} & x \end{bmatrix} \quad \textbf{(3.2.5.2.1.3)}$$

> $Eigenvalues(M)$

$$\begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \quad \textbf{(3.2.5.2.1.4)}$$

All eigenvectors satisfy

$$M \bullet V = \lambda \bullet V$$

where λ is an eigenvalue and V is an eigenvector, of the form

> $V := Vector([v_1, v_2, v_3])$

$$V := \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \textbf{(3.2.5.2.1.5)}$$

and the v_i, are the unknowns to be determined.

First eigenvector corresponding to the eigenvalue 0

> $M . V = 0 \cdot V$

$$\begin{bmatrix} -\mathrm{I}\sqrt{2}\, v_2 \\ \mathrm{I}\sqrt{2}\, v_1 - \mathrm{I}\sqrt{2}\, v_3 \\ \mathrm{I}\sqrt{2}\, v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{(3.2.5.2.1.6)}$$

> $lhs(\%) - rhs(\%)$

$$\begin{bmatrix} -\mathrm{I}\sqrt{2}\, v_2 \\ \mathrm{I}\sqrt{2}\, v_1 - \mathrm{I}\sqrt{2}\, v_3 \\ \mathrm{I}\sqrt{2}\, v_2 \end{bmatrix} \quad \textbf{(3.2.5.2.1.7)}$$

> $convert(\%, set)$

$$\left\{-\mathrm{I}\sqrt{2}\, v_2, \mathrm{I}\sqrt{2}\, v_2, \mathrm{I}\sqrt{2}\, v_1 - \mathrm{I}\sqrt{2}\, v_3\right\} \tag{3.2.5.2.1.8}$$

> $solve(\%)$

$$\left\{v_1 = v_3, v_2 = 0, v_3 = v_3\right\} \tag{3.2.5.2.1.9}$$

> $V_1 := subs(\%, V)$

$$V_1 := \begin{bmatrix} v_3 \\ 0 \\ v_3 \end{bmatrix} \tag{3.2.5.2.1.10}$$

For the second and third eigenvalues it is the same process, so copy the block of operations above and paste

> $M . V = 2 \cdot V$

$$\begin{bmatrix} -\mathrm{I}\sqrt{2}\, v_2 \\ \mathrm{I}\sqrt{2}\, v_1 - \mathrm{I}\sqrt{2}\, v_3 \\ \mathrm{I}\sqrt{2}\, v_2 \end{bmatrix} = \begin{bmatrix} 2\, v_1 \\ 2\, v_2 \\ 2\, v_3 \end{bmatrix} \tag{3.2.5.2.1.11}$$

> $lhs(\%) - rhs(\%)$

$$\begin{bmatrix} -\mathrm{I}\sqrt{2}\, v_2 - 2\, v_1 \\ \mathrm{I}\sqrt{2}\, v_1 - \mathrm{I}\sqrt{2}\, v_3 - 2\, v_2 \\ \mathrm{I}\sqrt{2}\, v_2 - 2\, v_3 \end{bmatrix} \tag{3.2.5.2.1.12}$$

> $convert(\%, set)$

$$\left\{-\mathrm{I}\sqrt{2}\, v_2 - 2\, v_1, \mathrm{I}\sqrt{2}\, v_2 - 2\, v_3, \mathrm{I}\sqrt{2}\, v_1 - \mathrm{I}\sqrt{2}\, v_3 - 2\, v_2\right\} \tag{3.2.5.2.1.13}$$

> $solve(\%)$

$$\left\{v_1 = -\frac{\mathrm{I}}{2}\sqrt{2}\, v_2, v_2 = v_2, v_3 = \frac{\mathrm{I}}{2}\sqrt{2}\, v_2\right\} \tag{3.2.5.2.1.14}$$

> $V_2 := subs(\%, V)$

$$V_2 := \begin{bmatrix} -\frac{\mathrm{I}}{2}\sqrt{2}\, v_2 \\ v_2 \\ \frac{\mathrm{I}}{2}\sqrt{2}\, v_2 \end{bmatrix} \tag{3.2.5.2.1.15}$$

Now for the third eigenvalue (again copy and paste)

> $M . V = -2 \cdot V$

$$\begin{bmatrix} -I\sqrt{2}\,v_2 \\ I\sqrt{2}\,v_1 - I\sqrt{2}\,v_3 \\ I\sqrt{2}\,v_2 \end{bmatrix} = \begin{bmatrix} -2\,v_1 \\ -2\,v_2 \\ -2\,v_3 \end{bmatrix} \quad \textbf{(3.2.5.2.1.16)}$$

> $lhs(\%) - rhs(\%)$

$$\begin{bmatrix} -I\sqrt{2}\,v_2 + 2\,v_1 \\ I\sqrt{2}\,v_1 - I\sqrt{2}\,v_3 + 2\,v_2 \\ I\sqrt{2}\,v_2 + 2\,v_3 \end{bmatrix} \quad \textbf{(3.2.5.2.1.17)}$$

> $convert(\%, set)$

$$\left\{-I\sqrt{2}\,v_2 + 2\,v_1, I\sqrt{2}\,v_2 + 2\,v_3, I\sqrt{2}\,v_1 - I\sqrt{2}\,v_3 + 2\,v_2\right\} \quad \textbf{(3.2.5.2.1.18)}$$

> $solve(\%)$

$$\left\{v_1 = \frac{I}{2}\sqrt{2}\,v_2, v_2 = v_2, v_3 = -\frac{I}{2}\sqrt{2}\,v_2\right\} \quad \textbf{(3.2.5.2.1.19)}$$

> $V_3 := subs(\%, V)$

$$V_3 := \begin{bmatrix} \frac{I}{2}\sqrt{2}\,v_2 \\ v_2 \\ -\frac{I}{2}\sqrt{2}\,v_2 \end{bmatrix} \quad \textbf{(3.2.5.2.1.20)}$$

So the three eigenvectors are

> V_1, V_2, V_3

$$\begin{bmatrix} v_3 \\ 0 \\ v_3 \end{bmatrix}, \begin{bmatrix} -\frac{I}{2}\sqrt{2}\,v_2 \\ v_2 \\ \frac{I}{2}\sqrt{2}\,v_2 \end{bmatrix}, \begin{bmatrix} \frac{I}{2}\sqrt{2}\,v_2 \\ v_2 \\ -\frac{I}{2}\sqrt{2}\,v_2 \end{bmatrix} \quad \textbf{(3.2.5.2.1.21)}$$

>

Advanced students: guiding them to program mathematical concepts on a computer algebra worksheet

D.T. Alves, J. V. Amaral, E.S. Cheb-Terrab and J.F. Medeiros Neto, “Learning Electromagnetism via Programming using Symbolic Computation”, Special issue of the Revista Brasileira de Ensino de Física (2002),

Status of the project

Prototypes of ***interfaces built*** cover:

• Natural numbers

• Functions

• Integer numbers

• Rational numbers

• Absolute value

• Logarithms

• Numerical sequences

• Trigonometry

• Matrices

• Determinants

• Linear systems

• Limits

• Derivatives

• Derivative of the inverse function

• The point in Cartesian coordinates

• The line

• The circle

• The ellipse

• The parabole

• The hyperbole

• The conics

More recent computer algebra frameworks: Maple Mobius for online courses and automated evaluation

The Maple system has a new framework for developing an integrated experience: *interactive interfaces* + *evaluation* + *databases of topics*, (concepts, problems and solutions)