Example 1. Consider the system (1.4) with $(d_1, d_2, d_3, r_1, r_2, r_3, a_{11}, a_{12}, a_{21}, a_{22}, a_{23}, a_{32}, a_{33}) = (1, 4, 1, 1, 1, 1, 0.5, 3, 2, 0.8, 1, 0.5, 0.9)$. Then the hypothesis (H1) is satisfied and hence (1.4) has only a positive constant steady state $E^* = (0.2529, 0.2912, 1.2729)$. By the computing software we have $B = -5.2900 < 0$. Consequently, (2.18) has a unique positive root $l_0 = 0.4704$ and thus we can obtain that $\omega_0 = 0.6904$, $\tau_0 = 0.4704$ and $h'(l_0) = 1.2769$. Lemma 2.1 together with Theorem 2.4 yield that E^* is locally asymptotically stable when $0 \leqslant \tau < \tau_0 = 0.4704$ and unstable when $\tau > \tau_0 = 0.4704$, and when τ passes increasingly through $\tau_0 = 0.4704$, a spatially homogeneous periodic solution emerges from E^*. In addition, we can compute $c_1(0) = -0.8724 - 1.6683i$. From the discussions in Section 4, we know that the spatially homogeneous bifurcating periodic solutions are stable on the center manifold. See Figs. 1 and 2.

$$\begin{cases} u_{1t} = d_1 u_{1xx} + u_1(t,x)[r_1 - a_{11}u_1(t,x) - a_{12}u_2(t,x)], \\ u_{2t} = d_2 u_{2xx} + u_2(t,x)[r_2 + a_{21}u_1(t-\tau,x) - a_{22}u_2(t,x) - a_{23}u_3(t-\tau,x)], \\ u_{3t} = d_3 u_{3xx} + u_3(t,x)[r_3 + a_{32}u_2(t,x) - a_{33}u_3(t,x)], \\ 0 < x < \pi, \ t > 0, \\ u_{jx}(t,0) = u_{jx}(t,\pi) = 0, \ t \geqslant 0, \\ u_j(s,x) = \eta_j(s,x), \ (s,x) \in [-\tau, 0] \times [0,\pi], \ j = 1,2,3, \end{cases} \tag{1.4}$$

where $\tau = \tau_1 + \tau_2$.

Theorem 2.4. *Assume that (H1), $B < 0, h'(l_0) \neq 0$ and $d_2 > M_5$ hold. Then*

(i) The positive constant steady state $E^*(u_1^*, u_2^*, u_3^*)$ of system (1.4) is locally asymptotically stable for $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$.
(ii) System (1.4) undergoes a spatially homogeneous Hopf bifurcation at the positive steady state $E^*(u_1^*, u_2^*, u_3^*)$ when $\tau = \tau_0^{(j)}$ $(j = 0, 1, \ldots)$, i.e., a family of spatially homogeneous periodic solutions bifurcating from E^* when τ crosses through the critical values $\tau_0^{(j)}$ $(j = 0, 1, \ldots)$.

Next, we discuss the effect of diffusion on spatially homogeneous Hopf bifurcation. Consider Eq. (2.17) again. Noting that $p_k > 0$ in (2.14), so, if there exist $k_0 \in \mathbb{N}$ such that $a_{0k_0} - b_{0k_0} < 0$ in (2.15), then Eq. (2.17) has a unique positive root. For a fixed $k_0 \in \mathbb{N}$ there exists a small constant $\epsilon_1 > 0$ such that $a_{0k_0} - b_{0k_0} < 0$ when $\max\{d_j\} < \epsilon_1$ $(j = 1, 2, 3)$ under the condition $B < 0$. Then, Eq. (2.17) has a unique positive root, denoted by l_{k_0}, and hence Eq. (2.12) has a unique positive root $\omega_{k_0} = \sqrt{l_{k_0}}$. By (2.11), we have

$$\cos \omega_{k_0}\tau = \frac{b_{1k_0}\omega_{k_0}^4 + (a_{2k_0}b_{0k_0} - a_{1k_0}b_{1k_0})\omega_{k_0}^2 - a_{0k_0}b_{0k_0}}{b_{0k_0}^2 + b_{1k_0}^2\omega_{k_0}^2}.$$

Thus, if we denote

$$\tau_{k_0}^{(j)} = \frac{1}{\omega_{k_0}}\left\{\cos^{-1}\left(\frac{b_{1k_0}\omega_{k_0}^4 + (a_{2k_0}b_{0k_0} - a_{1k_0}b_{1k_0})\omega_{k_0}^2 - a_{0k_0}b_{0k_0}}{b_{0k_0}^2 + b_{1k_0}^2\omega_{k_0}^2}\right) + 2j\pi\right\}, \quad j = 0, 1, \ldots, \tag{2.20}$$

then $\pm i\omega_{k_0}$ is a pair of purely imaginary roots of Eq. (2.7) when $\tau = \tau_{k_0}^{(j)}$. We assume that $h'(l_{k_0}) \neq 0$, where $h(l)$ is defined by (2.18). Using the similar argument as that in the proof of Lemma 2.3, the following transversality condition holds

$$\left[\frac{d(\mathrm{Re}\lambda(\tau))}{d\tau}\right]_{\tau=\tau_{k_0}^{(j)}} \neq 0.$$

Thus, we have the following result.

$$h(l) = l^3 + p_k l^2 + q_k l + r_k. \tag{2.18}$$

When $k = 0$, it is easy to obtain from (2.14) and (2.15) that

$$\begin{cases} p_0 = (u_1^* a_{11})^2 + (u_2^* a_{22})^2 + (u_3^* a_{33})^2 > 0, \\ a_{00} - b_{00} = u_1^* u_2^* u_3^* B. \end{cases}$$

If $B < 0$, then $a_{00} - b_{00} < 0$ and hence $r_0 < 0$. According to Descartes's rule of signs (see, for example, Appendix 2 in [29]), Eq. (2.17) has a unique positive root, denoted by l_0, and thus Eq. (2.12) has a unique positive root $\omega_0 = \sqrt{l_0}$. By (2.11), we have

$$\cos\omega_0\tau = \frac{b_{10}\omega_0^4 + (a_{20}b_{00} - a_{10}b_{10})\omega_0^2 - a_{00}b_{00}}{b_{00}^2 + b_{10}^2\omega_0^2}.$$

Thus, if we denote

$$\tau_0^{(j)} = \frac{1}{\omega_0}\left\{\cos^{-1}\left(\frac{b_{10}\omega_0^4 + (a_{20}b_{00} - a_{10}b_{10})\omega_0^2 - a_{00}b_{00}}{b_{00}^2 + b_{10}^2\omega_0^2}\right) + 2j\pi\right\}, \quad j = 0, 1, \ldots, \tag{2.19}$$

Lemma 2.1. *Assume that the condition (H1) holds. Then the positive constant steady state $E^*(u_1^*, u_2^*, u_3^*)$ of system (1.4) is locally asymptotically stable when $\tau = 0$.*

Next we discuss the effect of the delay τ on the stability of the trivial solution of (2.3). Assume that $i\omega$ $(\omega > 0)$ is a root of Eq. (2.7). Then ω should satisfy the following equation for some $k \in \mathbb{N}_0$

$$-i\omega^3 - a_{2k}\omega^2 + a_{1k}i\omega + a_{0k} + (b_{1k}i\omega + b_{0k})(\cos\omega\tau - i\sin\omega\tau) = 0, \tag{2.10}$$

which implies that

$$\begin{cases} a_{2k}\omega^2 - a_{0k} = b_{0k}\cos\omega\tau + b_{1k}\omega\sin\omega\tau, \\ -\omega^3 + a_{1k}\omega = b_{0k}\sin\omega\tau - b_{1k}\omega\cos\omega\tau. \end{cases} \tag{2.11}$$

From (2.11), we have

$$\omega^6 + (a_{2k}^2 - 2a_{1k})\omega^4 + (a_{1k}^2 - 2a_{0k}a_{2k} - b_{1k}^2)\omega^2 + a_{0k}^2 - b_{0k}^2 = 0. \tag{2.12}$$

Let $l = \omega^2$ and denote

$$p_k = a_{2k}^2 - 2a_{1k}, \quad q_k = a_{1k}^2 - 2a_{0k}a_{2k} - b_{1k}^2, \quad r_k = a_{0k}^2 - b_{0k}^2. \tag{2.13}$$

Then from (2.8), we know

$$\begin{cases} p_k = (d_1k^2 + u_1^*a_{11})^2 + (d_2k^2 + u_2^*a_{22})^2 + (d_3k^2 + u_3^*a_{33})^2 > 0, \\ q_k = \Big[(d_1k^2 + u_1^*a_{11})(d_3k^2 + u_3^*a_{33})\Big]^2 + \Big[(d_2k^2 + u_2^*a_{22})(d_3k^2 + u_3^*a_{33})\Big]^2 \\ \qquad + \Big[(d_1k^2 + u_1^*a_{11})(d_2k^2 + u_2^*a_{22})\Big]^2 - (u_1^*u_2^*a_{12}a_{21} + u_2^*u_3^*a_{23}a_{32})^2 \end{cases} \tag{2.14}$$

and

$$\begin{aligned} a_{0k} - b_{0k} = {} & d_1d_2d_3k^6 + (d_1d_2u_3^*a_{33} + d_1d_3u_2^*a_{22} + d_2d_3u_1^*a_{11})k^4 \\ & + \big[d_1u_2^*u_3^*(a_{22}a_{33} - a_{23}a_{32}) + d_2u_1^*u_3^*a_{11}a_{33} + d_3u_1^*u_2^*(a_{11}a_{22} - a_{12}a_{21})\big]k^2 + u_1^*u_2^*u_3^*B, \end{aligned} \tag{2.15}$$

where

$$B = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \tag{2.16}$$

Thus, Eq. (2.12) is reduced to

$$l^3 + p_kl^2 + q_kl + r_k = 0. \tag{2.17}$$

Denote

$$h(l) = l^3 + p_kl^2 + q_kl + r_k. \tag{2.18}$$

When $k = 0$, it is easy to obtain from (2.14) and (2.15) that

$$\begin{cases} p_0 = (u_1^*a_{11})^2 + (u_2^*a_{22})^2 + (u_3^*a_{33})^2 > 0, \\ a_{00} - b_{00} = u_1^*u_2^*u_3^*B. \end{cases}$$

If $B < 0$, then $a_{00} - b_{00} < 0$ and hence $r_0 < 0$. According to Descartes's rule of signs (see, for example, Appendix 2 in [29]), Eq. (2.17) has a unique positive root, denoted by l_0, and thus Eq. (2.12) has a unique positive root $\omega_0 = \sqrt{l_0}$. By (2.11), we have

$$\cos\omega_0\tau = \frac{b_{10}\omega_0^4 + (a_{20}b_{00} - a_{10}b_{10})\omega_0^2 - a_{00}b_{00}}{b_{00}^2 + b_{10}^2\omega_0^2}.$$

Thus, if we denote

$$\tau_0^{(j)} = \frac{1}{\omega_0}\left\{\cos^{-1}\left(\frac{b_{10}\omega_0^4 + (a_{20}b_{00} - a_{10}b_{10})\omega_0^2 - a_{00}b_{00}}{b_{00}^2 + b_{10}^2\omega_0^2}\right) + 2j\pi\right\}, \quad j = 0, 1, \ldots, \tag{2.19}$$

then $\pm i\omega_0$ is a pair of purely imaginary roots of Eq. (2.7) when $\tau = \tau_0^{(j)}$.

When $k \in \mathbb{N} = \{1, 2, \ldots\}$, if $a_{0k} - b_{0k} > 0$ (i.e., $r_k > 0$) in (2.15) and $q_k > 0$ in (2.14), then Eq. (2.17) has no positive root and hence Eq. (2.7) has no purely imaginary root. Under the condition $B < 0$ we can not determine the sign of $a_{22}a_{33} - a_{23}a_{32}$ and $a_{11}a_{22} - a_{12}a_{21}$ in (2.15). Therefore, we consider the following four cases:

(i) If $a_{22}a_{33} - a_{23}a_{32} \geqslant 0$, $a_{11}a_{22} - a_{12}a_{21} \geqslant 0$, then there exists a constant $M_1 > 0$ such that $a_{0k} - b_{0k} > 0$ and $q_k > 0$ when $\min\{d_j\} > M_1$ $(j = 1, 2, 3)$.
(ii) If $a_{22}a_{33} - a_{23}a_{32} < 0$, $a_{11}a_{22} - a_{12}a_{21} \geqslant 0$, then there exists a constant $M_2 > d_1$ such that $a_{0k} - b_{0k} > 0$ and $q_k > 0$ when $\min\{d_2, d_3\} > M_2$.
(iii) If $a_{22}a_{33} - a_{23}a_{32} \geqslant 0$, $a_{11}a_{22} - a_{12}a_{21} < 0$, then there exists a constant $M_3 > d_3$ such that $a_{0k} - b_{0k} > 0$ and $q_k > 0$ when $\min\{d_1, d_2\} > M_3$.
(iv) If $a_{22}a_{33} - a_{23}a_{32} < 0$, $a_{11}a_{22} - a_{12}a_{21} < 0$, then there exists a constant $M_4 > \max\{d_1, d_3\}$ such that $a_{0k} - b_{0k} > 0$ and $q_k > 0$ when $d_2 > M_4$.

The above analysis imply that there exists a constant $M_5 = \max\{M_j\}$ $(j = 1, 2, 3, 4)$ such that $a_{0k} - b_{0k} > 0$ and $q_k > 0$ when $d_2 > M_5$. Consequently, we can get the following conclusions.

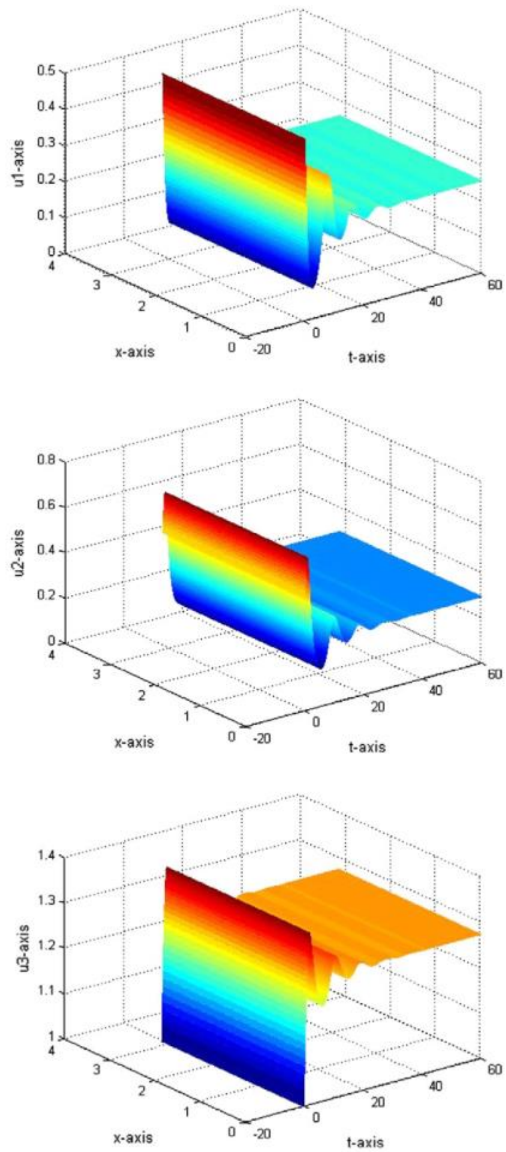

Fig. 1. The numerical simulations of system (1.4) with the parameters given in Example 1, $\tau = 0.3$ and initial conditions $u_1(t,x) = 0.5$, $u_2(t,x) = 0.5$, $u_3(t,x) = 1$, $(t,x) \in [-0.3, 0] \times [0, \pi]$. The positive constant steady state E^* is stable.

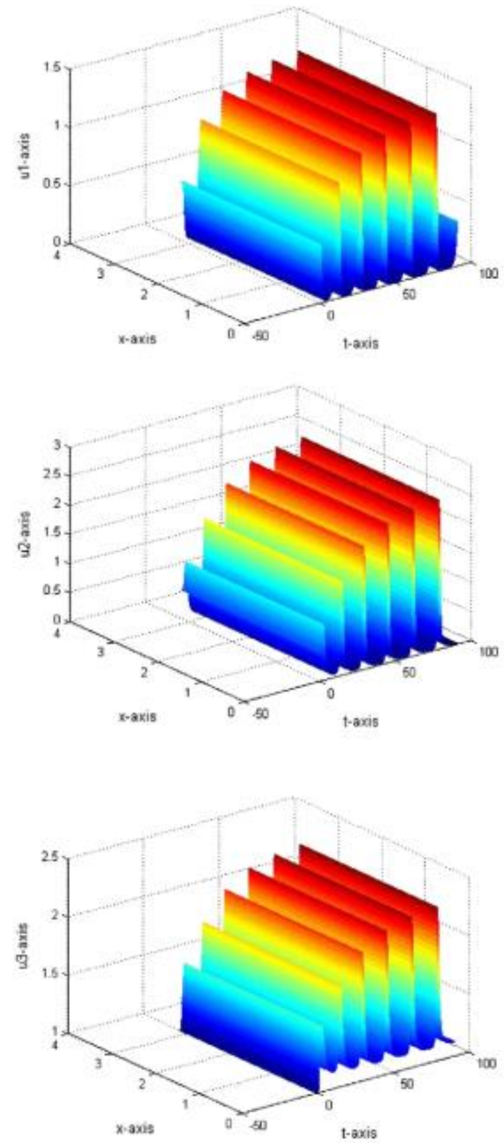

Fig. 2. The numerical simulations of system (1.4) with the parameters in Example 1, $\tau = 0.8$ and initial conditions $u_1(t,x) = 0.5$, $u_2(t,x) = 0.5$, $u_3(t,x) = 1$, $(t,x) \in [-0.8, 0] \times [0, \pi]$. The positive constant steady state E^* is unstable and system (1.4) bifurcates spatially homogeneous periodic solutions near the positive constant steady state.