Solving systems of nonlinear equations.
Description of the method.

Draghilev's method develops the idea of the method of continuation by parameter. Continuation by parameter assumes the use of a parameter as an independent variable, artificially introduced into the system or assigned from among the variables initially present in the system. Let us take as a basis one of the simplest variants of the method of continuation by parameter. Let (1) be a system of equations with respect to Y:

$$G(Y) = 0; \qquad (1)$$
$$G = (g_1, \dots, g_n); \quad g_i \in R^1; \; g_i = g_i(Y); \; Y = (y_1, \dots, y_n); \quad y_i \in R^1;$$

Let's choose a point

$$Y_0 = (y_{01}, \dots, y_{0n});$$

The value that system (1) will take at this point will be

$$G(Y_0) = G_0;$$

And let's consider a new system of equations:

$$G(Y) - \nu G_0 = 0; \qquad (1a)$$
$$\nu \in R^1;$$

In this system, there is a parameter ν that, in the case of $\nu = 0$, turns (1a) into (1), and in the case when ν it is equal to 1, then at the point

$$Y_0 = (y_{01}, ..., y_{0n});$$

the left side of (1a) becomes 0. Thus, if in (1a) we assume that:

$$y_i = y_i(v); \quad i = 1, ..., n; \quad v \in [0.1];$$

By changing the parameter v from 1 to 0, we can come from the solution (1a) to the solution (1). The section of the curve that (1a) describes must correspond to a monotonic change in all coordinates. This is a weaker condition than for Newton's method, but if we are talking about finding a solution almost "blindly", the parameter continuation method has few advantages over methods like Newton's method.

If in (1a) we consider n+1 coordinates, then (1a) will become a system of n equations with respect to n+ 1 variable. And when the value of the n+1-th variable v takes the value 0, then the remaining variables will deliver the solution to the original system of equations. The initial value v for 1, and the values of the remaining coordinates are chosen as the starting point for finding the solution to the system of nonlinear equations (1). In the process of constructing the curve, the sign of v may change, which will mean that the curve intersects the space of dimension n, and in this place there will be a region containing the solution (1). The number of solutions found will correspond to the number of changes in the sign of the parameter. The curve can pass through extreme points. This allows finding solutions very far from the initial point and finding more than one solution for one initial point. We can say that the method has a weak dependence on the initial approximation.

Let us consider a system of n equations in which the number of variables is n+1. Such a system describes a certain curve in a space of dimension n+1:

$$F(X) = 0; \qquad (2)$$
$$F = (f_1, ..., f_n),\ f_i \in R^1,\ f_i = f_i(X),\ X = (x_1, ..., x_{n+1}),\ \ x_j \in R^1;$$

We construct the curve (2), taking the arc length of the curve itself as the independent variable. The coordinates of the curve points are obtained as the solution of the Cauchy problem for a system of ordinary differential equations, the initial data for which will be the coordinates of a previously known point belonging to this curve:

$$
\begin{cases}
\frac{\partial f_1}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial f_1}{\partial x_{n+1}}\frac{dx_{n+1}}{dt} \\
. \\
. \qquad\qquad\qquad\qquad = 0; \\
. \\
\frac{\partial f_n}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial f_n}{\partial x_{n+1}}\frac{dx_{n+1}}{dt} \\
x_j(0) = x_{0j},\ j = 1,\ldots,n+1;
\end{cases} \qquad (3)
$$

(3) is a homogeneous system of linear equations n with respect to n+1 derivatives of the sought coordinates. We solve the system using Cramer's rule. Any derivative can be taken as a free variable, for example, the n+1 derivative. We assign it the value of the main determinant of the system to get rid of the denominators. And instead of (3) we get the following system of equations:

$$
\begin{cases}
\frac{dx_j}{dt} = \det\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_1}{\partial x_{n+1}} \ldots \frac{\partial f_1}{\partial x_n} \\ . & & . \\ . & (\text{ j-column}) & \\ . & & . \\ \frac{\partial f_n}{\partial x_1} & \ldots & \frac{\partial f_n}{\partial x_{n+1}} \ldots \frac{\partial f_n}{\partial x_n} \end{bmatrix},\ j = 1,..,n; \\
\frac{dx_{n+1}}{dt} = -\det\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_1}{\partial x_n} \\ . & & \\ . & & \\ . & & \\ \frac{\partial f_n}{\partial x_1} & \ldots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}; \\
x_j(0) = x_{0j},\ j = 1,\ldots,n+1;
\end{cases} \qquad (3a)
$$

The arc length does not depend on the choice of parameter t, and it increases with t. The value t=0 corresponds to the chosen starting point, from which the curve will be constructed in both directions, in the "positive" direction the integration step will be positive, and in the "negative" direction it will be negative. To solve (3a), we can apply the standard algorithm for the numerical solution of the Cauchy problem to construct the curve (2).

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