CIRCLES INSCRIBED IN A RIGHT TRIANGLE

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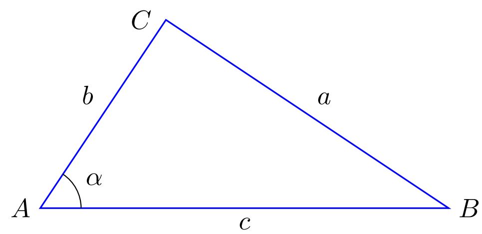
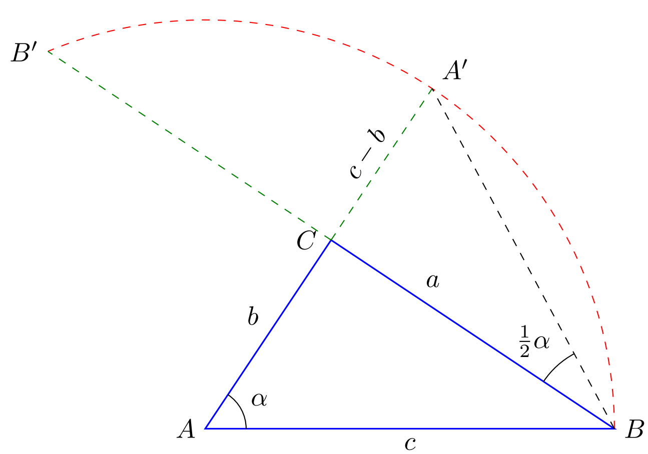
Theorem 1. *Consider the right triangle ABC with the right angle at vertex C, and side lengths a, b, and c, as shown in the diagram below. If α is the size of the angle at the vertex A, then we have*

$$\tan\frac{\alpha}{2} = \frac{c-b}{a}. \tag{1}$$



Proof. Draw the circular arc BB' centered at A as shown in the diagram below, and let A' be where the extension of AC meets the arc. The arc's radius is c, and consequently the length of CA' is $c-b$.

The size of the arc $A'B$ is α as it subtends the angle $A'AB$. The size of the arc $A'B'$ is also α by symmetry, and therefore the angle $A'B'B'$ which is subtended by that arc is $\frac{1}{2}\alpha$. The theorem's assertion follows from the geometry of the right triangle $A'CB$.



□

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Remark 1. The theorem above may be demonstrated purely through trigonometry, as follows. From $\cos\alpha = \frac{b}{c}$ and the half-angle formula we get

$$2\cos^2\frac{\alpha}{2} - 1 = \frac{b}{c},$$

whence $\cos^2\frac{\alpha}{2} = \frac{c+b}{2c}$, and therefore

$$1 + \tan^2\frac{\alpha}{2} = \frac{1}{\cos^2\frac{\alpha}{2}} = \frac{2c}{c+b}.$$

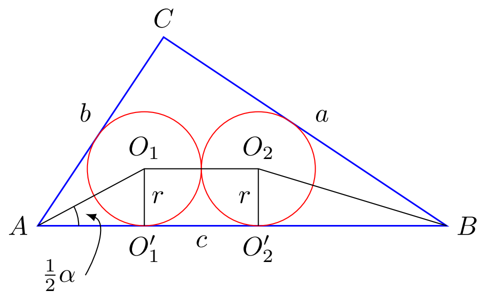
It follows that

$$\tan^2\frac{\alpha}{2} = \frac{2c}{c+b} - 1 = \frac{c-b}{c+b} = \frac{(c-b)^2}{(c-b)(c+b)} = \frac{(c-b)^2}{c^2-b^2} = \frac{(c-b)^2}{a^2},$$

whence $\tan\frac{\alpha}{2} = \frac{c-b}{a}$, as asserted.

Theorem 2. *Let ABC the right triangle as before, and suppose two mutually tangent circles of equal radii r have been inscribed in it, as shown below. Then*

$$r = \frac{c}{2 + \dfrac{a}{c-b} + \dfrac{b}{c-a}} \tag{2}$$



Proof. The center O_1 of the left circle lies on the bisector drawn from the vertex A. Therefore the angle O_1AO' is $\frac{1}{2}\alpha$ and consequently

$$AO_1' = \frac{r}{\tan\frac{\alpha}{2}} = \frac{r}{\frac{c-b}{a}} = \frac{a}{c-b}r,$$

where we have applied equation (1). Similarly,

$$O_2'B = \frac{b}{c-a}r.$$

Considering that $O_1'O_2' = 2r$, we obtain

$$\frac{a}{c-b}r + 2r + \frac{b}{c-a}r = c,$$

as asserted. □

Examples:

Equation (2) applied to the right triangle with the edge lengths $a = 3$, $b = 4$, $c = 5$ yields

$$r = \frac{5}{7}.$$

Similarly, for the right triangle with the edge lengths $a = 5$, $b = 12$, $c = 13$, we get

$$r = \frac{26}{17}.$$