CIRCLE ROLLING IN A ROLLING CIRCLE

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A circle of radius R and mass M rolls without slipping on a horizontal line L. A circle of smaller radius r and mass m rolls without slipping inside the larger circle. The motion takes place within a vertical plane. We wish to find the equations of motion of the system.

We set up the Cartesian xy coordinate system as follows. Consider the configuration of the system—the setup configuration—where the two circles touch the line L simultaneously. We take the common contact point as the origin of the coordinates, let the x axis run along the line L, and let the vertical y axis point *downward*; see Figure 1. On each circle we permanently mark their contact points with the x axis. These are shown as tiny gold-colored circles in the diagram. Actually we see only one circle in the setup configuration since the two contact points coincide. As the circles begin to move, the markers move along with them and become individually visible, as seen in Figure 2. We write C and c for the centers of the two circles;

Figure 2 shows the two circles at an arbitrary configuration during their motion. Note how the gold markers that started out at the origin have been displaced and now are shown as points A and B. As the large circle rolls, the radius CA makes an angle θ with respect to the vertical line CG. The no-slip condition implies that the length of the arc AG and the length of the line segment OG are equal. But the

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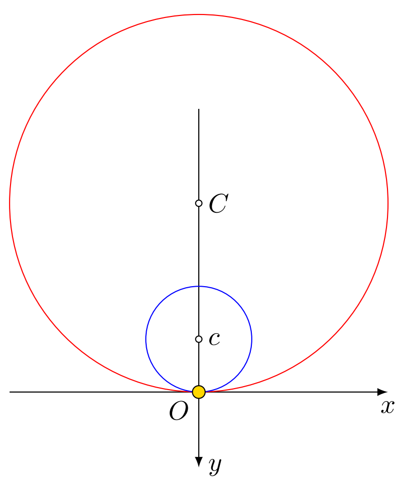

FIGURE 1. In this "setup configuration", the two circles touch the horizontal axis simultaneously. The contact point is marked on each circle through a gold-colored marker. As the circles move, they carry their resepective gold markers with them.

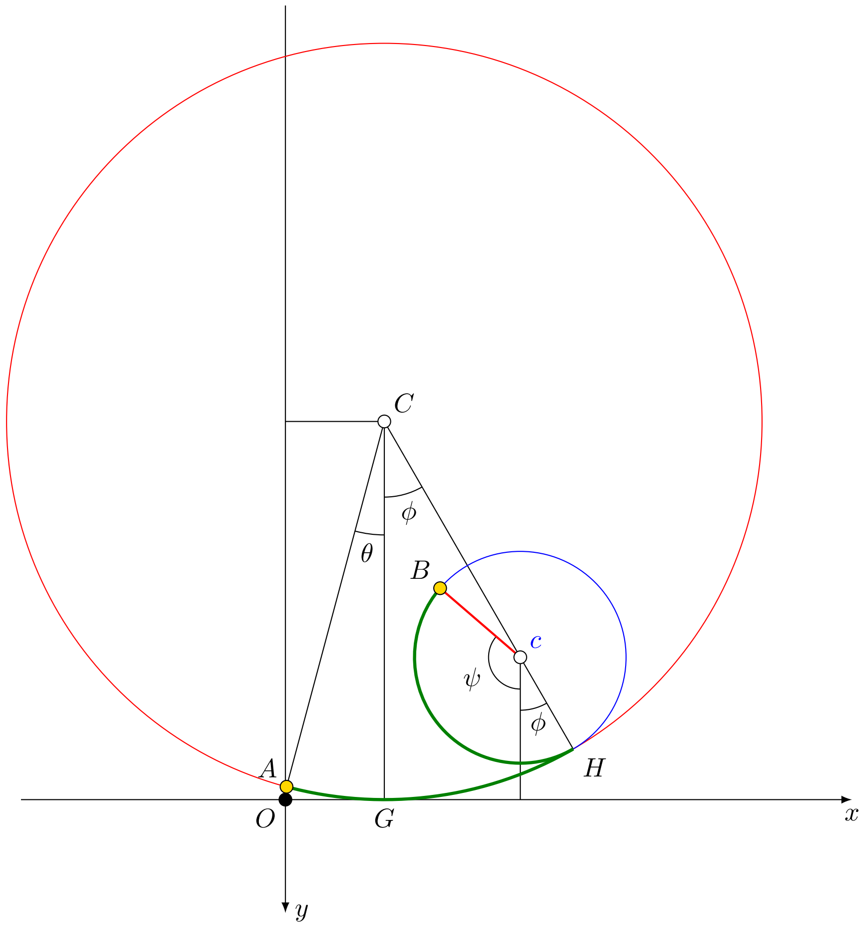

FIGURE 2. The no-slip condition on the small circle implies that the arcs HA and HB are of equal lengths.

length of AG is $R\theta$ and the length of OG is the x coordinate of C. Therefore the position vector of the center C is

$$\boldsymbol{C} = \begin{pmatrix} R\theta \\ -R \end{pmatrix} = R \begin{pmatrix} \theta \\ -1 \end{pmatrix}. \tag{1}$$

As to the small circle's gold marker B, let ϕ be the angle of the line segment Cc with respect to the vertical line CG. Let ψ be the angle of cB with respect to the vertical line through c. The no-slip condition on the small circle implies that the arcs HB and HA are of equal lengths. Therefore

$$R(\theta + \phi) = r(\theta + \psi). \tag{2}$$

The two-circle system has two degrees of freedom. We will use θ and ϕ for generalized coordinates. We use (2) to eliminate ψ in favor of θ and ϕ.

The position vector of the small circle's center c is

$$\boldsymbol{c} = \boldsymbol{C} + (R - r) \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} = R \begin{pmatrix} \theta \\ -1 \end{pmatrix} + (R - r) \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix}. \tag{3}$$

That's sufficient to construct the system's Lagrangian. We write J and j for the moments of inertia of the two circles about their centers. The kinetic energy is

$$T = \frac{1}{2}M|\dot{\boldsymbol{C}}|^2 + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m|\dot{\boldsymbol{c}}|^2 + \frac{1}{2}j\dot{\psi}^2,$$

and the potential energy is

$$V = -mgc_2,$$

where c_1 2s the y component of the position vector $\boldsymbol{c}$. Then the Lagrangian $L = T - V$ is is a function of ϕ and θ, and we are done.