Lemma. Let $\psi(t) = a(t + b \sin t)$, and let $\eta(t) = \psi(t) \bmod 2\pi$. We wish to show that η is periodic if and only if a is rational.

Part (a): Prove that η is periodic $\Rightarrow$ a is rational.

Suppose η is periodic with a period p, that is, $\eta(t + p) = \eta(t)$ for all t. Then $\psi(t + p) = \psi(t) + 2k\pi$ for some integer k. Thus

$$a\Big[(t + p) + b \sin(t + p)\Big] = a(t + b \sin t) + 2k\pi,$$

which simplifies to

$$ab\Big[\sin(t + p) - \sin(t)\Big] = 2k\pi - ap.$$

The right-hand side is a constant, therefore the left-hand side should also be a constant. That would be possible only if p is a multiple of 2π. Letting $p = 2m\pi$, the above simplifies to $2k\pi = ap$, that is $2k\pi = a(2m\pi)$, which leads to $a = \frac{k}{m}$, and thus a is rational, as asserted.

Part (b): Prove that a is rational $\Rightarrow$ η is periodic.

Suppose is some rational of the form $a = \frac{k}{m}$, where k and m are positive integers. We wish to show that η is periodic of period $p = 2m\pi$. So let's compute

$$\psi(t + 2m\pi) = a\Big[(t + 2m\pi) + b \sin(t + 2m\pi)\Big] = a\Big[t + 2m\pi + b \sin t\Big].$$

It follows that

$$\psi(t + 2m\pi) - \psi(t) = 2ma\pi = 2k\pi,$$

and therefore

$$\Big[\psi(t + 2m\pi) \bmod 2\pi\Big] - \Big[\psi(t) \bmod 2\pi\Big] = 2k\pi \bmod 2\pi = 0,$$

which shows that $\eta(t + 2m\pi) = \eta(t)$, as asserted.