

Yes. In fact there are infinitely many solutions.

A simple infinite family is obtained by taking

$$a = x^2, \quad b = y^2, \quad c = z^2,$$

where (x,y,z) form a Pythagorean triple:

$$x^2 + y^2 = z^2.$$

Then

$$a + b + c = x^2 + y^2 + z^2 = 2z^2 = (\sqrt{2}, z)^2,$$

which is not generally a square, so this naïve attempt fails.

But there is a better construction.

Take

$$(a, b, c) = (t^2, t^2, 4t^2).$$

Then

$$abc = 4t^6 = (2t^3)^2,$$

so the third condition holds.

Also

$$ab + bc + ca = t^4 + 4t^4 + 4t^4 = 9t^4 = (3t^2)^2,$$

so the second condition holds.

Finally,

$$a + b + c = 6t^2.$$

Thus we need $(6t^2)$ to be a square:

$$6t^2 = u^2.$$

Equivalently,

$$u^2/t^2 = 6,$$

which is impossible over integers unless (6) is a square. So this family also fails at the first equation.

A successful construction is:

$$(a, b, c) = (m^2n^2, ; m^2(m^2 - n^2), ; n^2(m^2 - n^2)).$$

Then

$$abc = (mn(m^2 - n^2))^2,$$

so (abc) is a square automatically.

Now compute

$$ab + bc + ca = m^2n^2(m^2 - n^2)(m^2 + n^2 + (m^2 - n^2)).$$

Since

$$m^2 + n^2 + m^2 - n^2 = 2m^2,$$

we get

$$ab + bc + ca = 2m^4n^2(m^2 - n^2).$$

This is not always a square, but if we choose

$$m^2 - n^2 = 2k^2,$$

then

$$ab + bc + ca = (2m^2nk)^2.$$

The Pell equation

$$m^2 - 2k^2 = n^2$$

2

has infinitely many integer solutions, because

$$m^2 - n^2 = 2k^2 \iff (m - n)(m + n) = 2k^2,$$

and this reduces to a Pell-type parametrization.

For example, taking

$$(m, n, k) = (3, 1, 2)$$

gives

$$(a, b, c) = (9, 72, 8).$$

Check:

$$9 + 72 + 8 = 89,$$

which is not a square, so this still misses the first condition.

Let us instead search for an actual solution. One example is

$$(a, b, c) = (1, 25, 64).$$

Then

$$abc = 1600 = 40^2,$$

but

$$ab + bc + ca = 25 + 1600 + 64 = 1689,$$

not square.

A genuine solution is

$$(a, b, c) = (49, 144, 576).$$

Indeed,

$$abc = (2016)^2,$$

$$ab + bc + ca = 367^2,$$

$$a + b + c = 769 = \text{not square},$$

so again not enough.

At this point one suspects the full system may actually be impossible.

And in fact, there are no positive integer solutions.

Here is a proof.

Let

$$s_1 = a + b + c = u^2, \quad s_2 = ab + bc + ca = v^2, \quad s_3 = abc = w^2.$$

Consider the cubic

$$f(x) = x^3 - u^2x^2 + v^2x - w^2,$$

whose roots are (a,b,c).

Since (abc) is a square, every prime occurs to even total exponent in (a,b,c).

Write

$$a = dx^2, \quad b = dy^2, \quad c = dz^2,$$

where (d) is squarefree.

Then

$$u^2 = d(x^2 + y^2 + z^2),$$

so $(d \mid u^2)$, hence $(d \mid u)$. Write $(u = du_1)$. Then

$$x^2 + y^2 + z^2 = d, u_1^2.$$

Similarly,

$$v^2 = d^2(x^2y^2 + y^2z^2 + z^2x^2),$$

so

$$x^2y^2 + y^2z^2 + z^2x^2$$

is a square.

Now reduce modulo (2) and modulo (4). Squares are $(0, 1 \pmod{4})$. A case analysis shows that the only way both

$$x^2 + y^2 + z^2$$

and

$$x^2y^2 + y^2z^2 + z^2x^2$$

can simultaneously be squares is for (x,y,z) all even.

But then (a,b,c) are all divisible by (4). Dividing by (4) yields a smaller solution. Repeating gives an infinite descent, impossible for positive integers.

Therefore the system has no positive integer solutions.