Physics Courseware Support: Mechanics

Hi
The attached worksheet is work in progress, to appear in Maple 2023 as Courseware support for Mechanics in the context of Physics courses. Everything below also works in Maple 2022.2 with the last Maplesoft Physics Updates for that release..

What follows is presented as "*Topic* > *Problem* > *Solution*", with typical symbolic problems and how you can solve them on a worksheet. As such, this material does not intend to compete with textbooks nor with teacher's notes but to be a helpful complement, as in "what can computer algebra really do to support the learning activity". Mainly, allow for focusing the logic and thinking while the computer takes care of the intricacies of the algebraic manipulations, that when computing with paper and pencil so frequently take mostly all of our focus.

The material, thus, has 70 solved problems covering all the sort-of-syllabus of hyperlinks below. The presentation uses notation as in textbooks and illustrates different techniques, several not present in help pages. It also shows why it is relevant to have a Vectors package that handles abstract vectors as well as projections using unit vectors, not matrix representations for them. Your feedback about everything you see in the worksheet - suggestions for new topics or problems, or anything else - can be useful and is welcome.

Due to the length of this material (~100 pages), out of the 70 problems, below I left open (visible) the Solution sections of only a few of them, illustrating different things, also new functionality e.g. the first and last ones. That is sufficient to have an idea of what this is about.

With the best wishes for 2023.

Explore. While learning, having success is a secondary goal: using your curiosity as a compass is what matters. Things can be done in many different ways, take full permission to make mistakes. Computer algebra can transform the algebraic computation part of physics into interesting discoveries and fun.

The following material assumes knowledge of how to use Maple. If you feel that is not your case, for a compact introduction on reproducing in Maple the computations you do with paper and pencil, see sections 1 to 5 of the *Mini-Course: Computer Algebra for Physicists*. *Also, the presentation assumes an understanding of the subjects and the style is not that of a textbook.* Instead, it focuses on conveniently using computer algebra to support the practice and learning process. The selection of topics follows references [1] *and* [2] *at the end. Maple 2023.0 includes Part I. Part II is forthcoming.*

Part I

Part II (forthcoming)

Position, velocity and acceleration in Cartesian, cylindrical and spherical coordinates

Load the *Physics:-Vectors* package

```
with(Physics:-Vectors)
```

$$[\&x, `+`, `.`, Assume, ChangeBasis, ChangeCoordinates, CompactDisplay, Component, Curl, DirectionalDiff, Divergence, Gradient, Identify, Laplacian, \nabla, Norm, ParametrizeCurve, ParametrizeSurface, ParametrizeVolume, Setup, Simplify, `\wedge`, diff, int] \quad (1)$$

Depending on the geometry of a problem, it can be convenient to work with either Cartesian or curvilinear coordinates. In an arbitrary reference system, the position in Cartesian coordinates and the basis of unitary vectors $(\hat{i}, \hat{j}, \hat{k})$ is given by

$r\_ = x\_i + y\_j + z\_k$

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \quad \textbf{(2)}$$

Problem

Rewrite the position vector $\vec{r}$ in cylindrical and spherical coordinates

Solution

To transform the position, start with cylindrical coordinates and the basis $(\hat{\rho}, \hat{\phi}, \hat{k})$, we need to use the transformation equations for both the Cartesian unit vectors and their related coordinates. For example

$\_i = ChangeBasis(\_i, cylindrical)$

$$\hat{i} = \cos(\phi)\,\hat{\rho} - \sin(\phi)\,\hat{\phi} \quad \textbf{(3)}$$

For all the unit vectors at once

$[\_i, \_j, \_k] =\sim ChangeBasis([\_i, \_j, \_k], cylindrical)$

$$\left[\hat{i} = \cos(\phi)\,\hat{\rho} - \sin(\phi)\,\hat{\phi}, \hat{j} = \sin(\phi)\,\hat{\rho} + \cos(\phi)\,\hat{\phi}, \hat{k} = \hat{k}\right] \quad \textbf{(4)}$$

The cylindrical basis shares the unit vector $\hat{k}$ with the Cartesian basis; also the corresponding coordinate z:

$[x, y, z] =\sim ChangeCoordinates([x, y, z], cylindrical)$

$$[x = \rho\cos(\phi), y = \rho\sin(\phi), z = z] \quad \textbf{(5)}$$

from which the position $\vec{r}$ in the cylindrical basis is given by

$subs(\textbf{(4)}, \textbf{(5)}, \textbf{(2)})$

$$\vec{r} = \rho\cos(\phi)\left(\cos(\phi)\,\hat{\rho} - \sin(\phi)\,\hat{\phi}\right) + \rho\sin(\phi)\left(\sin(\phi)\,\hat{\rho} + \cos(\phi)\,\hat{\phi}\right) + z\,\hat{k} \quad \textbf{(6)}$$

$simplify(\textbf{(6)})$

$$\vec{r} = z\,\hat{k} + \hat{\rho}\,\rho \quad \textbf{(7)}$$

For reference, also compute the inverse formulas

$[\_\rho, \_\phi, \_k] =\sim ChangeBasis([\_\rho, \_\phi, \_k], cartesian)$

$$\left[\hat{\rho} = \hat{i}\cos(\phi) + \sin(\phi)\,\hat{j}, \hat{\phi} = -\sin(\phi)\,\hat{i} + \cos(\phi)\,\hat{j}, \hat{k} = \hat{k}\right] \quad \textbf{(8)}$$

$[\rho, \phi, z] =\sim ChangeCoordinates([\rho, \phi, z], cartesian)$

$$\left[\rho = \sqrt{x^2 + y^2}, \phi = \arctan(y, x), z = z\right] \quad \textbf{(9)}$$

- To compute the position vector in spherical coordinates, or any other coordinate's system, the procedure is the same

$\_i = ChangeBasis(\_i, spherical)$

$$\hat{i} = \sin(\theta)\cos(\phi)\,\hat{r} + \cos(\theta)\cos(\phi)\,\hat{\theta} - \sin(\phi)\,\hat{\phi} \quad \textbf{(10)}$$

Note that the cylindrical and spherical systems share the coordinate ϕ and the corresponding unit vector $\hat{\phi}$. Relating the bases

$[\_i, \_j, \_k] =\sim ChangeBasis([\_i, \_j, \_k], spherical)$

$$\left[\hat{i} = \sin(\theta)\cos(\phi)\,\hat{r} + \cos(\theta)\cos(\phi)\,\hat{\theta} - \sin(\phi)\,\hat{\phi}, \hat{j} = \sin(\theta)\sin(\phi)\,\hat{r} + \cos(\theta)\sin(\phi)\,\hat{\theta} + \cos(\phi)\,\hat{\phi}, \hat{k} = \cos(\theta)\,\hat{r} - \sin(\theta)\,\hat{\theta}\right] \quad \textbf{(11)}$$

$[\_r, \_\theta, \_\phi] =\sim ChangeBasis([\_r, \_\theta, \_\phi], cartesian)$

$$\left[\hat{r} = \sin(\theta)\cos(\phi)\,\hat{i} + \sin(\theta)\sin(\phi)\,\hat{j} + \cos(\theta)\,\hat{k}, \hat{\theta} = \cos(\theta)\cos(\phi)\,\hat{i} + \cos(\theta)\sin(\phi)\,\hat{j} - \sin(\theta)\,\hat{k}, \hat{\phi} = -\sin(\phi)\,\hat{i} + \cos(\phi)\,\hat{j}\right] \quad \textbf{(12)}$$

Relating the coordinates

$[x, y, z] =\sim ChangeCoordinates([x, y, z], spherical)$

$$[x = r\sin(\theta)\cos(\phi), y = r\sin(\theta)\sin(\phi), z = r\cos(\theta)] \quad \mathbf{(13)}$$

$[r, \theta, \phi] =\sim ChangeCoordinates([r, \theta, \phi], cartesian)$

$$\left[r = \sqrt{x^2 + y^2 + z^2}, \theta = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right), \phi = \arctan(y, x)\right] \quad \mathbf{(14)}$$

from which the position in spherical coordinates is

$subs(\mathbf{(11)}, \mathbf{(13)}, \mathbf{(2)})$

$$\vec{r} = r\sin(\theta)\cos(\phi)\left(\sin(\theta)\cos(\phi)\hat{r} + \cos(\theta)\cos(\phi)\hat{\theta} - \sin(\phi)\hat{\phi}\right) + r\sin(\theta)\sin(\phi)\left(\sin(\theta)\sin(\phi)\hat{r} + \cos(\theta)\sin(\phi)\hat{\theta} + \cos(\phi)\hat{\phi}\right) + r\cos(\theta)\left(\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}\right) \quad \mathbf{(15)}$$

$simplify(\mathbf{(15)})$

$$\vec{r} = r\,\hat{r} \quad \mathbf{(16)}$$

In this result we see the position vector is just the product of the radial coordinate r and the radial unit vector $\hat{r}$.

- These results **(7)** and **(16)** for cylindrical and spherical coordinates can also be computed in a single step by operating over the entire vector at the same time. From **(2)**, we have

(2)

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \quad \mathbf{(17)}$$

$lhs(\mathbf{(2)}) = ChangeBasis(rhs(\mathbf{(2)}), cylindrical, alsocomponents)$

$$\vec{r} = z\,\hat{k} + \rho\,\hat{\rho} \quad \mathbf{(18)}$$

$lhs(\mathbf{(2)}) = ChangeBasis(rhs(\mathbf{(2)}), spherical, alsocomponents)$

$$\vec{r} = r\,\hat{r} \quad \mathbf{(19)}$$

Starting from the position in the Cartesian system, now as functions of the time to allow for differentiation, first note that the Cartesian unit vectors $(\hat{i}, \hat{j}, \hat{k})$ *do not depend on time,* they are *constant* vectors. So $\vec{r}(t)$ is entered as

```
restart;
with(Physics:-Vectors) :
```

$r\_(t) = x(t)\,\_i + y(t)\,\_j + z(t)\,\_k$

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k} \quad \mathbf{(20)}$$

Before proceeding further, use a compact display to more clearly visualize the following expressions. When in doubt about the contents behind a given display, input *show* as shown below.

$CompactDisplay((x, y, z, \rho, r, \theta, \phi, \_\rho, \_r, \_\theta, \_\phi)(t))$

x(t) will now be displayed as x

y(t) will now be displayed as y

$z(t)$ *will now be displayed as* z

$\rho(t)$ *will now be displayed as* ρ

$r(t)$ *will now be displayed as* r

$\theta(t)$ *will now be displayed as* θ

$\phi(t)$ *will now be displayed as* ϕ

$\hat{\rho}(t)$ *will now be displayed as* $\hat{\rho}$

$\hat{r}(t)$ *will now be displayed as* $\hat{r}$

$\hat{\theta}(t)$ *will now be displayed as* $\hat{\theta}$

$\hat{\phi}(t)$ *will now be displayed as* $\hat{\phi}$ **(21)**

For the velocity and acceleration, note the dot notation for derivatives with respect to t

$v\_(t) = diff(rhs(\mathbf{(20)}), t)$

$$\vec{v}(t) = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k} \quad \mathbf{(22)}$$

show

$$\vec{v}(t) = \dot{x}(t)\,\hat{i} + \dot{y}(t)\,\hat{j} + \dot{z}(t)\,\hat{k} \quad \mathbf{(23)}$$

$a\_(t) = diff(rhs(\mathbf{(22)}), t)$

$$\vec{a}(t) = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} + \ddot{z}\,\hat{k} \quad \mathbf{(24)}$$

The position $\vec{r}(t)$ as a function of time

Problem

Given the position vector as a function of the time *t*, rewrite it in cylindrical and spherical coordinates while making the curvilinear unit vectors' time dependency explicit.

Solution

To compute the velocity and acceleration in curvilinear coordinates, the procedure for cylindrical, spherical, or any other curvilinear system of coordinates is always the same. Start by recomputing the position as a function of time

$lhs(\mathbf{(20)}) = ChangeBasis(rhs(\mathbf{(20)}), cylindrical, also)$

** Partial match of 'also' against keyword 'alsothecoordinates'*

$$\vec{r}(t) = z\,\hat{k} + \hat{\rho}\,\rho \quad \mathbf{(25)}$$

show

$$\vec{r}(t) = z(t)\,\hat{k} + \hat{\rho}(t)\,\rho(t) \quad \mathbf{(26)}$$

We see that, unlike the case of the Cartesian unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, *the curvilinear unit vectors - in this case* $\hat{\rho}(t)$ *- depend on the time.* This is because their orientation changes along the trajectory $\vec{r}(t)$. The same happens when expressing the position vector in spherical coordinates

$lhs(\mathbf{(20)}) = ChangeBasis(rhs(\mathbf{(20)}), spherical, alsothecoordinates)$

$$\vec{r}(t) = r\,\hat{r} \quad \mathbf{(27)}$$

show

$$\vec{r}(t) = r(t)\,\hat{r}(t) \quad \mathbf{(28)}$$

The velocity $\vec{v}(t)$

Problem

Rewrite the velocity $\vec{v}(t) = \dot{\vec{r}}(t)$ in cylindrical and spherical coordinates while making the curvilinear unit vectors' time dependency explicit .

Solution

For the velocity,
(22)

$$\vec{v}(t) = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k} \quad \mathbf{(29)}$$

$lhs(\mathbf{(22)}) = ChangeBasis(rhs(\mathbf{(22)}), cylindrical, alsothecoordinates)$

$$\vec{v}(t) = \dot{z}\,\hat{k} + \hat{\phi}\,\rho\,\dot{\phi} + \hat{\rho}\,\dot{\rho} \quad \mathbf{(30)}$$

$lhs(\mathbf{(22)}) = ChangeBasis(rhs(\mathbf{(22)}), spherical, alsothecoordinates)$

$$\vec{v}(t) = r\,\dot{\phi}\,\hat{\phi}\,\sin(\theta) + r\,\dot{\theta}\,\hat{\theta} + \hat{r}\,\dot{r} \quad \mathbf{(31)}$$

show

$$\vec{v}(t) = r(t)\,\dot{\phi}(t)\,\hat{\phi}(t)\,\sin(\theta(t)) + r(t)\,\dot{\theta}(t)\,\hat{\theta}(t) + \hat{r}(t)\,\dot{r}(t) \quad \mathbf{(32)}$$

This result in spherical coordinates - as well as the results **(30)** in cylindrical coordinates - can also be achieved by directly differentiating the position in the corresponding coordinates, e.g. for spherical:
$\mathbf{(27)} \equiv \vec{r}(t) = r\,\hat{r}$
$diff(\mathbf{(27)}, t)$

$$\dot{\vec{r}}(t) = r\,\dot{\phi}\,\hat{\phi}\,\sin(\theta) + r\,\dot{\theta}\,\hat{\theta} + \hat{r}\,\dot{r} \quad \mathbf{(33)}$$

The acceleration $\vec{a}(t)$

Problem

Rewrite the acceleration $\vec{a}(t) = \ddot{\vec{r}}(t)$ in cylindrical and spherical components while making the curvilinear unit vectors' time dependency explicit.

Solution

For the acceleration, starting from **(24)**
(24)

$$\vec{a}(t) = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} + \ddot{z}\,\hat{k} \quad \mathbf{(34)}$$

$lhs((\mathbf{24})) = ChangeBasis(rhs((\mathbf{24})), cylindrical, alsothecoordinates)$

$$\vec{a}(t) = \left(-\rho\,\dot{\phi}^2 + \ddot{\rho}\right)\hat{\rho} + \left(\rho\,\ddot{\phi} + 2\,\dot{\rho}\,\dot{\phi}\right)\hat{\phi} + \ddot{z}\,\hat{k} \tag{35}$$

$lhs((\mathbf{24})) = ChangeBasis(rhs((\mathbf{24})), spherical, alsothecoordinates)$

$$\vec{a}(t) = \left(\ddot{r} + \left(r\cos(\theta)^2 - r\right)\dot{\phi}^2 - r\,\dot{\theta}^2\right)\hat{r} + \left(-r\sin(\theta)\cos(\theta)\,\dot{\phi}^2 + r\,\ddot{\theta} + 2\,\dot{\theta}\,\dot{r}\right)\hat{\theta} + \left(2\,r\cos(\theta)\,\dot{\phi}\,\dot{\theta} + r\,\ddot{\phi}\sin(\theta) + 2\sin(\theta)\,\dot{\phi}\,\dot{r}\right)\hat{\phi} \tag{36}$$

This result in spherical coordinates - as well as the results **(35)** in cylindrical coordinates - can also be achieved by directly differentiating the position in the corresponding coordinates, e.g. for spherical:

(27) $\equiv \vec{r}(t) = r\,\hat{r}$

$diff((\mathbf{27}), t, t)$

$$\ddot{\vec{r}}(t) = \hat{r}\left(\ddot{r} - r\,\dot{\phi}^2\sin(\theta)^2 - r\,\dot{\theta}^2\right) + \left(-r\sin(\theta)\cos(\theta)\,\dot{\phi}^2 + r\,\ddot{\theta} + 2\,\dot{\theta}\,\dot{r}\right)\hat{\theta} + \left(2\,r\cos(\theta)\,\dot{\phi}\,\dot{\theta} + r\,\ddot{\phi}\sin(\theta) + 2\sin(\theta)\,\dot{\phi}\,\dot{r}\right)\hat{\phi} \tag{37}$$

Deriving these formulas

All these results for the position $\vec{r}$, velocity $\vec{v}$ and acceleration $\vec{a}$ are based on the differentiation rules for cylindrical and spherical unit vectors. It is thus instructive to also be able to derive any of these formulas; for that, we need the differentiation rule for the unit vectors. For example, for the spherical ones

```
restart;
with(Physics:-Vectors) :
CompactDisplay((x, y, z, ρ, r, θ, ϕ, _ρ, _r, _θ, _ϕ)(t), quiet)
```

```
map(%diff = diff, [_r, _θ, _ϕ](t) , t)
```

$$\left[\frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta} + \dot{\phi}\sin(\theta)\,\hat{\phi},\ \frac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r} + \dot{\phi}\cos(\theta)\,\hat{\phi},\ \frac{d\hat{\phi}}{dt} = -\dot{\phi}\,\hat{\rho}\right] \tag{38}$$

The above result contains, in the last equation, the cylindrical radial unit vector $\hat{\rho}(t)$; rewrite it in the spherical basis

$\_\rho(t) = ChangeBasis(\_\rho(t), spherical)$

$$\hat{\rho} = \sin(\theta)\,\hat{r} + \cos(\theta)\,\hat{\theta} \tag{39}$$

So the differentiation rules for spherical unit vectors, with the result expressed in the spherical system, are

$subs((\mathbf{39}), (\mathbf{38}))$

$$\left[\frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta} + \dot{\phi}\sin(\theta)\,\hat{\phi},\ \frac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r} + \dot{\phi}\cos(\theta)\,\hat{\phi},\ \frac{d\hat{\phi}}{dt} = -\dot{\phi}\left(\sin(\theta)\,\hat{r} + \cos(\theta)\,\hat{\theta}\right)\right] \tag{40}$$

Problem

With this information at hand, derive, in steps, the expressions for the velocity and acceleration in cylindrical and spherical coordinates

Solution

We want to compute
%diff((**27**), *t*)

$$\frac{\partial}{\partial t}\left(\vec{r}(t)=r\,\hat{r}\right) \tag{41}$$

expand((**41**))

$$\frac{d}{dt}\vec{r}(t)=\left(\frac{dr}{dt}\right)\hat{r}+r\left(\frac{d\hat{r}}{dt}\right) \tag{42}$$

Introducing the differentiation rules **(40)** for the unit vectors
subs((**40**), (**42**))

$$\frac{d}{dt}\vec{r}(t)=\left(\frac{dr}{dt}\right)\hat{r}+r\left(\dot{\theta}\,\hat{\theta}+\dot{\phi}\sin(\theta)\,\hat{\phi}\right) \tag{43}$$

Performing the *inert* (grayed) derivatives
value((**43**))

$$\dot{\vec{r}}(t)=\dot{r}\,\hat{r}+r\left(\dot{\theta}\,\hat{\theta}+\dot{\phi}\sin(\theta)\,\hat{\phi}\right) \tag{44}$$

In the same way, for the acceleration
%diff((**27**), *t*, *t*)

$$\frac{\partial^2}{\partial t^2}\left(\vec{r}(t)=r\,\hat{r}\right) \tag{45}$$

expand((**45**))

$$\frac{d^2}{dt^2}\vec{r}(t)=\left(\frac{d^2r}{dt^2}\right)\hat{r}+2\left(\frac{dr}{dt}\right)\left(\frac{d\hat{r}}{dt}\right)+r\left(\frac{d^2\hat{r}}{dt^2}\right) \tag{46}$$

subs((**40**), (**46**))

$$\frac{d^2}{dt^2}\vec{r}(t)=\left(\frac{d^2r}{dt^2}\right)\hat{r}+2\left(\frac{dr}{dt}\right)\left(\dot{\theta}\,\hat{\theta}+\dot{\phi}\sin(\theta)\,\hat{\phi}\right)+r\left(\frac{\partial}{\partial t}\left(\dot{\theta}\,\hat{\theta}+\dot{\phi}\sin(\theta)\,\hat{\phi}\right)\right) \tag{47}$$

expand((**47**))

$$\frac{d^2}{dt^2}\vec{r}(t)=\left(\frac{d^2r}{dt^2}\right)\hat{r}+2\left(\frac{dr}{dt}\right)\dot{\theta}\,\hat{\theta}+2\left(\frac{dr}{dt}\right)\dot{\phi}\sin(\theta)\,\hat{\phi}+r\left(\frac{d^2\theta}{dt^2}\right)\hat{\theta}+r\,\dot{\theta}\left(\frac{d\hat{\theta}}{dt}\right)+r\left(\frac{d^2\phi}{dt^2}\right)\sin(\theta)\,\hat{\phi}+r\,\dot{\phi}\left(\frac{d\theta}{dt}\right)\cos(\theta)\,\hat{\phi}+r\,\dot{\phi}\sin(\theta)\left(\frac{d\hat{\phi}}{dt}\right) \tag{48}$$

subs((**40**), (**48**))

$$\frac{d^2}{dt^2}\vec{r}(t)=\left(\frac{d^2r}{dt^2}\right)\hat{r}+2\left(\frac{dr}{dt}\right)\dot{\theta}\,\hat{\theta}+2\left(\frac{dr}{dt}\right)\dot{\phi}\sin(\theta)\,\hat{\phi}+r\left(\frac{d^2\theta}{dt^2}\right)\hat{\theta}+r\,\dot{\theta}\left(-\dot{\theta}\,\hat{r}+\dot{\phi}\cos(\theta)\,\hat{\phi}\right)+r\left(\frac{d^2\phi}{dt^2}\right)\sin(\theta)\,\hat{\phi}+r\,\dot{\phi}\left(\frac{d\theta}{dt}\right)\cos(\theta)\,\hat{\phi}-r\,\dot{\phi}^2\sin(\theta)\left(\sin(\theta)\,\hat{r}+\cos(\theta)\,\hat{\theta}\right) \tag{49}$$

value((**49**))

$$\ddot{\vec{r}}(t) = \ddot{r}\,\hat{r} + 2\,\dot{r}\,\dot{\theta}\,\hat{\theta} + 2\,\dot{r}\,\dot{\phi}\sin(\theta)\,\hat{\phi} + r\,\ddot{\theta}\,\hat{\theta} + r\,\dot{\theta}\left(-\dot{\theta}\,\hat{r} + \dot{\phi}\cos(\theta)\,\hat{\phi}\right) + r\,\ddot{\phi}\sin(\theta)\,\hat{\phi} + r\,\dot{\phi}\,\dot{\theta}\cos(\theta)\,\hat{\phi} - r\,\dot{\phi}^2\sin(\theta)\left(\sin(\theta)\,\hat{r} + \cos(\theta)\,\hat{\theta}\right) \quad \textbf{(50)}$$

Collect vector components

Physics:-*Vectors*:-*Collect*(**(50)**)

$$\ddot{\vec{r}}(t) = \left(2\,r\,\dot{\theta}\,\dot{\phi}\cos(\theta) + 2\,\dot{r}\,\dot{\phi}\sin(\theta) + r\,\ddot{\phi}\sin(\theta)\right)\hat{\phi} + \left(-r\,\dot{\phi}^2\sin(\theta)^2 - r\,\dot{\theta}^2 + \ddot{r}\right)\hat{r} + \left(-\dot{\phi}^2\sin(\theta)\,r\cos(\theta) + 2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)\hat{\theta} \quad \textbf{(51)}$$

Summary

- You can express $\vec{r}(t)$, $\vec{v}(t)$ and $\vec{a}(t)$ in any of the Cartesian, cylindrical or spherical systems via three different methods: 1) using the *ChangeBasis* command 2) differentiating 3) deriving the formulas by differentiating in steps, starting from the differentiation rules for the curvilinear unit vectors.

Velocity and acceleration in the case of 2-dimensional motion on the x, y plane

Problem

Derive formulas for velocity and acceleration in the case of 2-dimensional motion on the x, y plane, starting from the general 3-dimensional formulas above, e.g. **(44)** and **(51)** in spherical coordinates. Specialize the resulting formulas for the case of *circular motion*.

Solution

Starting from the general 3-dimensional formula **(44)** in spherical coordinates, the formulas for 2-dimensional motion on the x, y plane can be obtained taking $\theta = \frac{\pi}{2}$. For the velocity, we have

$eval\left(\textbf{(44)}, \theta(t) = \frac{\pi}{2}\right)$

$$\dot{\vec{r}}(t) = \dot{r}\,\hat{r} + r\,\dot{\phi}\,\hat{\phi} \quad \textbf{(52)}$$

Note that while this formula involves the coordinate r, since by construction $\theta(t) = \frac{\pi}{2}$, the coordinate r on the (x, y) plane is equal to the cylindrical radial coordinate ρ; indeed, **(52)** is equal to the formula **(30)** if we take $z(t) = 0$, $\dot{z}(t) = 0$,

$\%eval(\textbf{(30)}, z(t) = 0)$

$$\left.\left(\vec{v}(t) = \dot{z}\,\hat{k} + \hat{\phi}\,\rho\,\dot{\phi} + \hat{\rho}\,\dot{\rho}\right)\right|_{z = 0} \quad \textbf{(53)}$$

$value(\textbf{(53)})$

$$\vec{v}(t) = \hat{\phi}\,\rho\,\dot{\phi} + \hat{\rho}\,\dot{\rho} \quad \textbf{(54)}$$

Also, **(52)** is written in terms of the radial coordinate r, but in view of $\theta(t) = \frac{\pi}{2}$, the radial coordinate lies on the plane, so again we have $r = \rho$.

For the acceleration,

$eval\left(\mathbf{(51)}, \theta(t) = \frac{\pi}{2}\right)$

$$\ddot{\vec{r}}(t) = (2\,\dot{r}\,\dot{\phi} + r\,\ddot{\phi})\,\hat{\phi} + \left(-r\,\dot{\phi}^2 + \ddot{r}\right)\hat{r} \quad \mathbf{(55)}$$

$\%eval(\mathbf{(35)}, z(t) = 0)$

$$\left(\vec{a}(t) = \left(-\rho\,\dot{\phi}^2 + \ddot{\rho}\right)\hat{\rho} + (\rho\,\ddot{\phi} + 2\,\dot{\rho}\,\dot{\phi})\,\hat{\phi} + \ddot{z}\,\hat{k}\right)\Big|_{z=0} \quad \mathbf{(56)}$$

$value(\mathbf{(56)})$

$$\vec{a}(t) = \left(-\rho\,\dot{\phi}^2 + \ddot{\rho}\right)\hat{\rho} + (\rho\,\ddot{\phi} + 2\,\dot{\rho}\,\dot{\phi})\,\hat{\phi} \quad \mathbf{(57)}$$

- The particular case of ***circular motion*** on the x, y plane is obtained by taking $r(t) = r_0$, a constant in addition to $\theta = \frac{\pi}{2}$,

$eval\left(\mathbf{(44)}, \left[\theta(t) = \frac{\pi}{2}, r(t) = r_0\right]\right)$

$$\dot{\vec{r}}(t) = r_0\,\dot{\phi}\,\hat{\phi} \quad \mathbf{(58)}$$

$eval\left(\mathbf{(51)}, \left[\theta(t) = \frac{\pi}{2}, r(t) = r_0\right]\right)$

$$\ddot{\vec{r}}(t) = r_0\,\ddot{\phi}\,\hat{\phi} - r_0\,\dot{\phi}^2\,\hat{r} \quad \mathbf{(59)}$$

Here too we have $\hat{r} = \hat{\rho}$ since $\theta(t) = \frac{\pi}{2}$, and we recover the expression of the acceleration in cylindrical coordinates

$\%eval\left(\mathbf{(57)}, \rho(t) = r_0\right)$

$$\left(\vec{a}(t) = \left(-\rho\,\dot{\phi}^2 + \ddot{\rho}\right)\hat{\rho} + (\rho\,\ddot{\phi} + 2\,\dot{\rho}\,\dot{\phi})\,\hat{\phi}\right)\Big|_{\rho = r_0} \quad \mathbf{(60)}$$

$value(\mathbf{(60)})$

$$\vec{a}(t) = -r_0\,\dot{\phi}^2\,\hat{\rho} + r_0\,\ddot{\phi}\,\hat{\phi} \quad \mathbf{(61)}$$

The equations of motion

A single particle

restart;

with(*Physics*:-*Vectors*) :
CompactDisplay((*r_*, *p_*, *F_*, *L_*, *N_*)(*t*))

$$\vec{r}(t) \text{ will now be displayed as } \vec{r}$$

$$\vec{p}(t) \text{ will now be displayed as } \vec{p}$$

$$\vec{F}(t) \text{ will now be displayed as } \vec{F}$$

$$\vec{L}(t) \text{ will now be displayed as } \vec{L}$$

$$\vec{N}(t) \text{ will now be displayed as } \vec{N} \quad \textbf{(62)}$$

The equation of motion of a single particle is Newton's 2^{nd} law
*F_*(*t*) = *m*·*diff*(*r_*(*t*), *t*, *t*)

$$\vec{F} = m\, \ddot{\vec{r}} \quad \textbf{(63)}$$

where $\ddot{\vec{r}}(t) = \vec{a}(t)$ is the acceleration and $m\, \dot{\vec{r}}(t) = \vec{p}(t)$ is the linear momentum, so in terms of $\vec{p}$
*F_*(*t*) = *diff*(*p_*(*t*), *t*)

$$\vec{F} = \dot{\vec{p}} \quad \textbf{(64)}$$

We define the angular momentum $\vec{L}$ of a particle, and the torque $\vec{N}$ acting upon it, as
*L_*(*t*) = *r_*(*t*) × *p_*(*t*)

$$\vec{L} = \vec{r} \times \vec{p} \quad \textbf{(65)}$$

*N_*(*t*) = *r_*(*t*) × *F_*(*t*)

$$\vec{N} = \vec{r} \times \vec{F} \quad \textbf{(66)}$$

Differentiating the definition of $\vec{L}$
diff(**(65)**, *t*)

$$\dot{\vec{L}} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} \quad \textbf{(67)}$$

Since $\dot{\vec{r}} = \vec{v}$ is parallel to $\vec{p} = m\, \vec{v}$, the first term in the above cancels, and in the second term, from **(64)**, $\dot{\vec{p}} = \vec{F}$
eval(**(67)**, [*diff*(*r_*(*t*), *t*) = 0, *diff*(*p_*(*t*), *t*) = *F_*(*t*)])

$$\dot{\vec{L}} = \vec{r} \times \vec{F} \quad \textbf{(68)}$$

from which
subs((*rhs* = *lhs*)(**(68)**), **(66)**)

$$\vec{N} = \dot{\vec{L}} \quad \textbf{(69)}$$

- As discussed below, in the case of a closed system, $\vec{F} = 0$ and these two equations result in

$$\dot{\vec{p}} = 0, \quad \dot{\vec{L}} = 0$$

that is, the linear and angular momentum are conserved quantities. Note that $\dot{\vec{L}} = 0$ does not require that $\vec{F} = 0$, only that $\vec{r} \times \vec{F} = 0$.

The equations of motion - vectorial form

Problem
Assuming that the acceleration is known as a function of *t*, compute:
a) The trajectory $\vec{r}(t)$ starting from $\vec{a}(t) = \ddot{\vec{r}}(t)$
b) A solution for each of the three Cartesian components
c) A solution for generic initial conditions

Solution

restart;
with(*Physics*:-*Vectors*) :
CompactDisplay((*x*, *y*, *z*, ρ, *r*, θ, ϕ, _ρ, _r, _θ, _ϕ)(*t*), *quiet*) :

a) Let $\vec{r}(t)$ be the position of the particle in a reference system; then, the velocity and acceleration are given by
$v\_(t) = diff(r\_(t), t)$

$$\vec{v}(t) = \dot{\vec{r}}(t) \tag{70}$$

$a\_(t) = diff(r\_(t), t, t)$

$$\vec{a}(t) = \ddot{\vec{r}}(t) \tag{71}$$

If the acceleration is known as a function of *t*, the trajectory is computed by integrating **(71)**
dsolve(**(71)**)

$$\vec{r}(t) = \int \left(\int \vec{a}(t)\, \mathrm{d}t \right) \mathrm{d}t + \vec{c}_1\, t + \vec{c}_2 \tag{72}$$

where the vectorial integration constants, $\vec{c}_1$ and $\vec{c}_2$, are specified by the initial conditions of the problem (see **c)** below), typically by the position and velocity at some instant, say $\vec{r}(t)\Big|_{t = t_0} = \vec{r}_0$ and $\dot{\vec{r}}(t)\Big|_{t = t_0} = \vec{v}_0$.

---

b) The integration of vectorial equations is also frequently performed after expressing $\vec{r}(t)$, $\vec{v}(t)$ and $\vec{a}(t)$ in a particular system of coordinates. For example, in the Cartesian system **(71)** has the form
(*rhs* = *lhs*)(**(24)**)

$$\ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} + \ddot{z}\,\hat{k} = \vec{a}(t) \tag{73}$$

Now suppose that the three components of the acceleration are known as a function of time
$subs\left(a\_(t) = a_x(t) \cdot \_i + a_y(t) \cdot \_j + a_z(t) \cdot \_k, \mathbf{(73)}\right)$

$$\ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} + \ddot{z}\,\hat{k} = a_x(t)\,\hat{i} + a_y(t)\,\hat{j} + a_z(t)\,\hat{k} \tag{74}$$

Vectorial equations like this one can be integrated directly, provided that they are expressed in a particular system of coordinates and the unit vectors are constant or known expressions of the time
dsolve(**(74)**)

$$\hat{i}\,x + \hat{j}\,y + \hat{k}\,z = \hat{i}\left(\int \left(\int a_x(t)\,\mathrm{d}t\right)\mathrm{d}t + c_3\,t + c_4\right) + \hat{j}\left(\int \left(\int a_y(t)\,\mathrm{d}t\right)\mathrm{d}t + c_5\,t + c_6\right) + \hat{k}\left(\int \left(\int a_z(t)\,\mathrm{d}t\right)\mathrm{d}t + c_7\,t + c_8\right) \tag{75}$$

---

c) The vectorial initial conditions $\vec{r}_0$ and $\vec{v}_0$, specifying the integration constants $\{c_3, c_4, c_5, c_6, c_7, c_8\}$, can also be written in components

$x(t_0) = x_0, y(t_0) = y_0, z(t_0) = z_0(t), diff(x(t_0), t_0) = v_{x0}, diff(y(t_0), t_0) = v_{y0}, diff(z(t_0), t_0) = v_{z0}$

$$x(t_0) = x_0, y(t_0) = y_0, z(t_0) = z_0(t), x_{t_0} = v_{x0}, y_{t_0} = v_{y0}, z_{t_0} = v_{z0} \quad \textbf{(76)}$$

Passing this information, the system can be solved taking these initial conditions into account -

$dsolve([\textbf{(74)}, \textbf{(76)}], \{x, y, z\}(t))$

$$\left\{x = \int_{t_0}^{t} \left(\int_{t_0}^{\tau} a_x(\tau)\, d\tau\right) d\tau + v_{x0}\, t - t_0\, v_{x0} + x_0, y = \int_{t_0}^{t} \left(\int_{t_0}^{\tau} a_y(\tau)\, d\tau\right) d\tau + v_{y0}\, t - t_0\, v_{y0} + y_0, z = \int_{t_0}^{t} \left(\int_{t_0}^{\tau} a_z(\tau)\, d\tau\right) d\tau + v_{z0}\, t + c_4\right\} \quad \textbf{(77)}$$

Note that a vectorial equation is also always equivalent to a system of equations, one for each of the components, with or without initial conditions:

$convert(\textbf{(74)}, setofequations)$

$$\{\ddot{x} = a_x(t), \ddot{y} = a_y(t), \ddot{z} = a_z(t)\} \quad \textbf{(78)}$$

$dsolve(\textbf{(78)}, \{x, y, z\})$

$$\left\{x = \int \left(\int a_x(t)\, dt\right) dt + c_7\, t + c_8, y = \int \left(\int a_y(t)\, dt\right) dt + c_5\, t + c_6, z = \int \left(\int a_z(t)\, dt\right) dt + c_3\, t + c_4\right\} \quad \textbf{(79)}$$

The case of constant acceleration

Problem

Starting from the vectorial equation **(72)** for $\vec{r}(t)$, derive the formula for constant acceleration

Solution

restart;
with(*Physics*:-*Vectors*) :

From the vectorial equation
(72)

$$\vec{r}(t) = \int \left(\int \vec{a}(t)\, dt\right) dt + \vec{c}_1\, t + \vec{c}_2 \quad \textbf{(80)}$$

in the particular case of constant acceleration,

$eval(\textbf{(72)}, a\_(t) = a_{0\_})$

$$\vec{r}(t) = \int \left(\int \vec{a}_0\, dt\right) dt + \vec{c}_1\, t + \vec{c}_2 \quad \textbf{(81)}$$

$value(\textbf{(81)})$

$$\vec{r}(t) = \frac{1}{2}\vec{a}_0 t^2 + \vec{c}_1 t + \vec{c}_2 \quad \mathbf{(82)}$$

The two vectorial arbitrary constants $\vec{c}_1$ and $\vec{c}_2$ can be specialized after giving initial conditions, the position and velocity at some point in time, which in the most general form are entered as

$\%eval\left(r\_(t), t = t_0\right) = r_{0\_}, \%eval\left(diff(r\_(t), t), t = t_0\right) = v_{0\_}$

$$\vec{r}(t)\Big|_{t = t_0} = \vec{r}_0, \dot{\vec{r}}(t)\Big|_{t = t_0} = \vec{v}_0 \quad \mathbf{(83)}$$

The vectorial differential equation **(71)** $\equiv \vec{a} = \ddot{\vec{r}}$

$eval\left(\mathbf{(71)}, a\_(t) = a_{0\_}\right)$

$$\vec{a}_0 = \ddot{\vec{r}}(t) \quad \mathbf{(84)}$$

Its solution, using the initial conditions **(83)** (the ordering in the list passed to dsolve is irrelevant), is the traditional high-school formula

$dsolve([\mathbf{(83)}, \mathbf{(84)}], r\_(t))$

$$\vec{r}(t) = \frac{\left(t - t_0\right)^2 \vec{a}_0}{2} + \left(t - t_0\right)\vec{v}_0 + \vec{r}_0 \quad \mathbf{(85)}$$

Motion under gravitational force close to the Earth's surface

Problem

Derive a formula for motion under gravitational force close to the Earth's surface

Solution

restart;
with(*Physics*:-*Vectors*) :

In a reference system whose origin is at the center of the Earth, the gravitational acceleration of a small (if compared with the planet) mass that is *close to the surface* (if compared with the planet's radius) is approximately constant in magnitude

$g\_ = -g\ \_r$

$$\vec{g} = -g\,\hat{r} \quad \mathbf{(86)}$$

where $\hat{r}$ is the radial unit vector that points outwards. Placing a reference system on the surface of the planet, thus not far away from the system's origin, the surface can be approximated to "flat" and the radial direction of the gravitational acceleration can be approximated to *constant vertical*, i.e. $\vec{g} = -g\,\hat{k}$. If, in addition, we consider only a reasonably small interval of time, such that the reference system can be considered *inertial* (the rotation of the planet in such a small amount of time can be neglected), we have

$a_{0\_} = -g\ \_k$

$$\vec{a}_0 = -g\,\hat{k} \quad \mathbf{(87)}$$

and for the sake of simplicity we chose the initial value $t_0 = 0$, so the trajectory **(85)** of a particle

(85)

$$\vec{r}(t) = \frac{(t-t_0)^2 \vec{a}_0}{2} + (t-t_0)\vec{v}_0 + \vec{r}_0 \qquad \textbf{(88)}$$

becomes

$subs\left(\textbf{(87)}, t_0 = 0, \textbf{(85)}\right)$

$$\vec{r}(t) = -\frac{1}{2} t^2 g \hat{k} + t \vec{v}_0 + \vec{r}_0 \qquad \textbf{(89)}$$

Motion under gravitational force not close to the Earth's surface

The problem of two particles of masses m_1 and m_2 gravitationally attracted to each other, discarding relativistic effects, is formulated by Newton's law of gravity: the particles attract each other - so both move - with a force along the line that joins the particles and whose magnitude is proportional to $\frac{1}{r^2}$, where r represents the distance between the particles (this problem is treated in general form in the more advanced sections).

Problem

As a specific case, consider the problem of a particle of mass $m \ll M$, where M is earth's mass, moving ***not close to the surface*** (if compared with the radius of earth).

Solution

restart;
with(*Physics*:-*Vectors*) :
CompactDisplay(($x, y, z, \rho, r, \theta, \phi$, _$\rho$, _$r$, _$\theta$, _$\phi$, r_)(t), *quiet*)

The movement can be approximated to that of a particle of mass m (which is much smaller than M) moving under a central force in a reference system with its origin at the center of the planet. The gravitational force acting on m is thus

$F\_(t) = -\frac{G\, M\, m\, \_r(t)}{r(t)^2}$

$$\vec{F}(t) = -\frac{G M m \hat{r}}{r^2} \qquad \textbf{(90)}$$

where G is the gravitational constant and we are assuming that the planet is not moving (because $M \gg m$). Here $r(t)$ is the distance from center of the planet to the particle, that is the radial spherical coordinate r, and $\hat{r}(t)$ is the radial unit vector. The minus sign reflects the fact that the particle of small mass m is attracted to the center of the planet of big mass M.

The vectorial equation of motion is thus

$\frac{rhs(\textbf{(90)})}{m} = diff(r\_(t), t, t)$

$$-\frac{G M \hat{r}}{r^2} = \ddot{\vec{r}} \qquad \textbf{(91)}$$

Since the force has its simpler form expressed in the spherical system, it is appropriate to also express the acceleration on the right-hand side in the spherical coordinates and basis using **(37)**

(37)

$$\ddot{\vec{r}} = \hat{r}\left(\ddot{r} - r\dot{\phi}^2 \sin(\theta)^2 - r\dot{\theta}^2\right) + \left(-r\sin(\theta)\cos(\theta)\dot{\phi}^2 + r\ddot{\theta} + 2\dot{\theta}\dot{r}\right)\hat{\theta} + \left(2r\cos(\theta)\dot{\phi}\dot{\theta} + r\ddot{\phi}\sin(\theta) + 2\sin(\theta)\dot{\phi}\dot{r}\right)\hat{\phi} \quad \textbf{(92)}$$

It can be shown that the motion happens on a plane. So by orienting the z axis perpendicular to that plane we can directly simplify this formulation taking $\theta(t) = \frac{\pi}{2}$

$eval\left(\textbf{(37)}, \theta(t) = \frac{\pi}{2}\right)$

$$\ddot{\vec{r}} = \hat{r}\left(\ddot{r} - r\dot{\phi}^2\right) + \left(r\ddot{\phi} + 2\dot{\phi}\dot{r}\right)\hat{\phi} \quad \textbf{(93)}$$

Substituting into **(91)** we get

$subs(\textbf{(93)}, \textbf{(91)})$

$$-\frac{G M \hat{r}}{r^2} = \hat{r}\left(\ddot{r} - r\dot{\phi}^2\right) + \left(r\ddot{\phi} + 2\dot{\phi}\dot{r}\right)\hat{\phi} \quad \textbf{(94)}$$

Rewriting this vectorial equation as a system of equations,

$convert(\textbf{(94)}, setofequations)$

$$\left\{0 = r\ddot{\phi} + 2\dot{\phi}\dot{r}, -\frac{G M}{r^2} = \ddot{r} - r\dot{\phi}^2\right\} \quad \textbf{(95)}$$

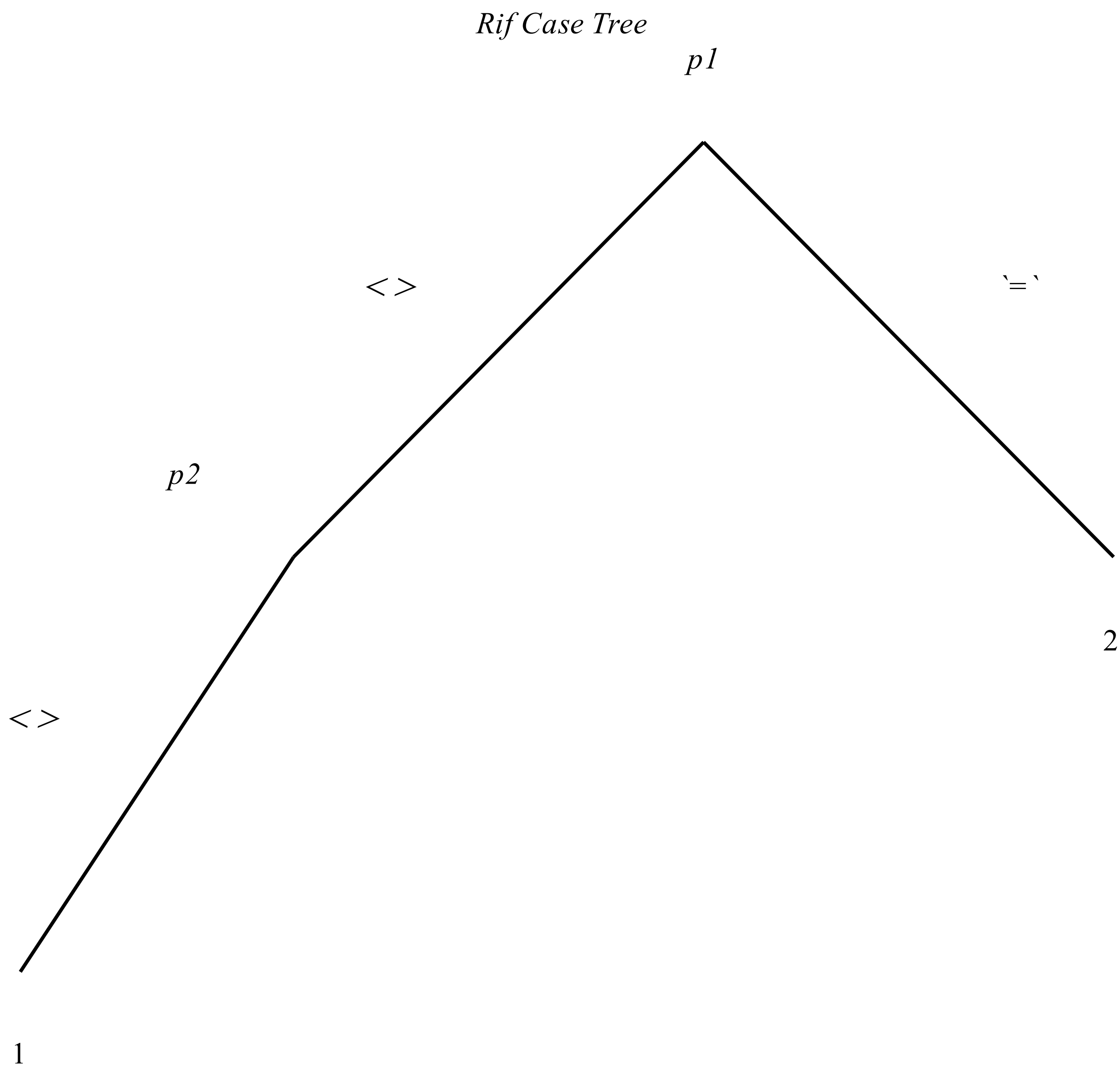
This is a non-linear coupled system of equations for the unknowns $\phi(t)$ and $r(t)$. This problem is further discussed in the section Motion in a central field, but, in general, for coupled systems of equations, it is instructive to first analyze the system decoupling it. To accomplish this we can use PDEtools:-casesplit. There are two possible orderings. With $[r, \phi]$, the system splits into two cases

$PDEtools{:}\text{-}casesplit(\textbf{(95)}, [r, \phi], caseplot)$

========= *Pivots Legend* =========

$$p1 = \dot{\phi}$$

$$p2 = 4\dot{\phi}^4 - 3\ddot{\phi}^2 + 2\dot{\phi}\dddot{\phi}$$



$$\left[r^3 = \frac{4\dot{\phi}^2 G M}{4\dot{\phi}^4 - 3\ddot{\phi}^2 + 2\dot{\phi}\dddot{\phi}}, \ddddot{\phi} = \frac{-4\ddot{\phi}\dot{\phi}^4 - 21\ddot{\phi}^3 + 22\ddot{\phi}\dot{\phi}\dddot{\phi}}{4\dot{\phi}^2}\right] \textbf{where } [r \neq 0, \dot{\phi} \neq 0], \left[\ddot{r} = -\frac{G M}{r^2}, \dot{\phi} = 0\right] \textbf{where } [r \neq 0] \quad \textbf{(96)}$$

The first case happens when $\dot{\phi} \neq 0$; the system is decoupled as a fourth order ODE for $\phi(t)$, and, assuming that ODE can be solved, $r(t)^3$ is then expressed as a function of that solution $\phi(t)$. With the opposite ordering the system decouples into a single case

PDEtools:-*casesplit*(**(95)**, $[\phi, r]$)

$$\left[\ddot{\phi} = -\frac{2\,\dot{\phi}\,\dot{r}}{r},\ \dot{\phi}^2 = \frac{\ddot{r}\,r^2 + G\,M}{r^3},\ \dddot{r} = \frac{-3\,\ddot{r}\,\dot{r}\,r^2 - \dot{r}\,G\,M}{r^3}\right] \textbf{ where } [r \neq 0] \tag{97}$$

This decoupling can be solved to the end, although the algebraic structure of the solution is of little use

dsolve(**(97)**)

$$\left[\left[\left\{r = RootOf\left(-\left(\int^{\_Z} \frac{\sqrt{\_a}}{RootOf\left(-\ln(\_a) - 2\left(\int^{\_Z} \frac{\_h}{\sqrt{4\,G^2\,M^2 - 4\,G\,M\,\_h^2 + \_h^4 + 4\,c_2}}\,\mathrm{d}\_h\right) + c_3\right)}\,\mathrm{d}\_a\right) + t + c_4\right),\ r = RootOf\left(-\left(\int^{\_Z} \frac{\sqrt{\_a}}{RootOf\left(-\ln(\_a) + 2\left(\int^{\_Z} \frac{\_h}{\sqrt{4\,G^2\,M^2 - 4\,G\,M\,\_h^2 + \_h^4 + 4\,c_2}}\,\mathrm{d}\_h\right) + c_3\right)}\,\mathrm{d}\_a\right) + t + c_4\right)\right\},\ \left\{\phi = \int \frac{\sqrt{r\,(\ddot{r}\,r^2 + G\,M)}}{r^2}\,\mathrm{d}t + c_1,\ \phi = \int -\frac{\sqrt{r\,(\ddot{r}\,r^2 + G\,M)}}{r^2}\,\mathrm{d}t + c_1\right\}\right]\right] \tag{98}$$

Circular motion

Problem

Determine the angular velocity $\dot{\phi}$ in the case of circular motion and show it is constant.

Solution

restart;
with(*Physics*:-*Vectors*) :
CompactDisplay((*x*, *y*, *z*, ρ, *r*, θ, ϕ, *_ρ*, *_r*, *_θ*, *_ϕ*) (*t*), *quiet*)

The equations of movement **(95)**
(95)

$$\left\{0 = r\,\ddot{\phi} + 2\,\dot{\phi}\,\dot{r}, -\frac{G\,M}{r^2} = \ddot{r} - r\,\dot{\phi}^2\right\} \tag{99}$$

are tractable in the case of *circular motion*, that is when $\dot{r} = 0$. In this case $r(t) = r_0$, where r_0 is a constant, and the starting system of equations is
eval(**(95)**, $r(t) = r_0$)

$$\left\{0 = r_0\,\ddot{\phi}, -\frac{G\,M}{r_0^2} = -r_0\,\dot{\phi}^2\right\} \tag{100}$$

The first of these equations shows that $r(t) = r_0$ automatically implies that $\ddot{\phi} = 0$, that the motion is circular, *and* that the motion has a constant angular velocity $\dot{\phi}$, which is given by the second equation
op(2, **(100)**)

$$-\frac{G\,M}{r_0^2} = -r_0\,\dot{\phi}^2 \tag{101}$$

This expression **(101)** is also called centripetal acceleration, and the value of the angular velocity $\dot{\phi}$ is
solve(**(101)**, {*diff*(ϕ(*t*), *t*)})

$$\left\{\dot{\phi} = \frac{\sqrt{r_0\,G\,M}}{r_0^2}\right\}, \left\{\dot{\phi} = -\frac{\sqrt{r_0\,G\,M}}{r_0^2}\right\} \tag{102}$$

Escape velocity

Problem
Determine the velocity that a particle of mass *m* should have at Earth's surface in order to escape the planet's gravitational attraction.

Solution

restart;
with(*Physics*:-*Vectors*) :
CompactDisplay((*x*, *y*, *z*, ρ, *r*, θ, ϕ, *_ρ*, *_r*, *_θ*, *_ϕ*) (*t*), *quiet*)

The escape velocity can be computed from the second case in **(96)**,
(96)

$$\left[r^3 = \frac{4\,\dot{\phi}^2\,G\,M}{4\,\dot{\phi}^4 - 3\,\ddot{\phi}^2 + 2\,\dot{\phi}\,\dddot{\phi}}, \ddddot{\phi} = \frac{-4\,\ddot{\phi}\,\dot{\phi}^4 - 21\,\ddot{\phi}^3 + 22\,\ddot{\phi}\,\dot{\phi}\,\dddot{\phi}}{4\,\dot{\phi}^2}\right] \textbf{where } [r \neq 0, \dot{\phi} \neq 0], \left[\ddot{r} = \right. \tag{103}$$

$$-\frac{G\,M}{r^2}, \dot{\phi} = 0 \Bigg| \textbf{where } [r \neq 0]$$

that is, when $\dot{\phi} = 0$. This case represents the particle of mass m either falling into the center of the planet following a straight line, or, depending on the initial value of the radial velocity, the particle is escaping from the attraction, arriving *at infinity* with radial velocity $v\Big|_{r=\infty} = 0$. So,

(96)[2]

$$\left[\ddot{r} = -\frac{G\,M}{r^2}, \dot{\phi} = 0\right] \textbf{where } [r \neq 0] \tag{104}$$

To compute the escape velocity, we need to express this equation in terms of the velocity

subs(*diff*(*r*(*t*), *t*) = *v*(*t*), **(104)**)

$$\left[\dot{v}(t) = -\frac{G\,M}{r^2}, \dot{\phi} = 0\right] \textbf{where } [r \neq 0] \tag{105}$$

then the velocity as a function of the position instead of time, as in

OFF

diff(*v*(*t*), *t*) = *diff*(*v*(*r*), *r*)·*v*(*r*)

$$\dot{v}(t) = \left(\frac{d}{dr}\, v(r)\right) v(r) \tag{106}$$

and also remove r's dependency on the time, so take $r(t) = r$

subs(**(106)**, *r*(*t*) = *r*, **(105)**)

$$\left[\left(\frac{d}{dr}\, v(r)\right) v(r) = -\frac{G\,M}{r^2}, \dot{\phi}(t) = 0\right] \textbf{where } [r \neq 0] \tag{107}$$

Take the contents of this structure

op(*op*(1, **(107)**))

$$\left(\frac{d}{dr}\, v(r)\right) v(r) = -\frac{G\,M}{r^2}, \dot{\phi}(t) = 0 \tag{108}$$

Suppose the initial velocity is the velocity when the small particle of mass m is at the surface of the planet where $r = R$, the radius of the planet. So $v(R) = v_0$. We want to calculate the value of v_0 such that $v(\infty) = 0$. For that purpose, solve the system **(108)** with that initial value of the velocity; discarding the equation $\dot{\phi} = 0$ as unnecessary, we are left with the system we want to solve

[**(108)**[1], *v*(*R*) = v_0]

$$\left[\left(\frac{d}{dr}\, v(r)\right) v(r) = -\frac{G\,M}{r^2}, v(R) = v_0\right] \tag{109}$$

All the symbols involved are greater than 0, so tackle the problem *assuming positive*

dsolve(**(109)**) assuming *positive*

$$v(r) = \frac{\sqrt{R\,r\left(R\,r\,v_0^2 + 2\,G\,M\,R - 2\,G\,M\,r\right)}}{r\,R} \tag{110}$$

The condition is that $v\Big|_{r=\infty} = 0$

0 = *limit*(*rhs*(**(110)**), *r* = infinity)

$$0 = \frac{\sqrt{R^2 v_0^2 - 2 G M R}}{R} \quad \textbf{(111)}$$

From which the escape velocity we are looking for is
$solve\left(\textbf{(111)}, \{v_0\}\right)$

$$\left\{v_0 = \frac{\sqrt{2}\,\sqrt{G M R}}{R}\right\}, \left\{v_0 = -\frac{\sqrt{2}\,\sqrt{G M R}}{R}\right\} \quad \textbf{(112)}$$

Different acceleration in different regions

Problem
Suppose a particle is moving along the x axis according to

$$x(t) = t^3 - 8 t^2 + 18 t + 3$$

a) Determine the regions where the motion has positive and negative acceleration. Compute the position at $t \to \infty$.

b) Compute the velocity $v_x(t)$ corresponding to $x(t) = t^3 - 8 t^2 + 18 t + 3$, starting - not from this expression for $x(t)$ but from the acceleration $a_x(t) = \ddot{x}(t)$

Solution

restart;
with(*Physics*:-*Vectors*) :

$x(t) = t^3 - 8 t^2 + 18 t + 3$

$$x(t) = t^3 - 8 t^2 + 18 t + 3 \quad \textbf{(113)}$$

a) The acceleration is the second derivative of the position. In this case $y(t) = 0$, $z(t) = 0$, so
$a_x(t) = diff(rhs(\textbf{(113)}), t, t)$

$$a_x(t) = 6 t - 16 \quad \textbf{(114)}$$

Therefore, the motion is *retarded* (negative acceleration) when
$6 t - 16 < 0$

$$6 t < 16 \quad \textbf{(115)}$$

that is, the open interval (to represent intervals, see RealRange)
$solve(\textbf{(115)}, t)$

$$\left(-\infty, \frac{8}{3}\right) \quad \textbf{(116)}$$

The motion has positive acceleration in the interval
$solve(6 t - 16 > 0, t)$

$$\left(\frac{8}{3}, \infty\right) \quad \textbf{(117)}$$

When $t \to \infty$, the position x approaches ∞ as expected, since the particle moves in a region of positive

and increasing acceleration
$limit(\mathbf{(113)}, t = \text{infinity})$

$$\lim_{t \to \infty} x(t) = \infty \tag{118}$$

b) The acceleration $a_x(t) = \ddot{x}(t)$ is
$diff(v_x(t), t) = rhs(\mathbf{(114)})$

$$\dot{v}_x(t) = 6\,t - 16 \tag{119}$$

$dsolve(\mathbf{(119)})$

$$v_x(t) = 3\,t^2 + c_1 - 16\,t \tag{120}$$

The integration constant c_3 needs to be adjusted to match the given trajectory **(106)**. To do that,
$diff(x(t), t) = rhs(\mathbf{(120)})$

$$\dot{x}(t) = 3\,t^2 + c_1 - 16\,t \tag{121}$$

$dsolve(\mathbf{(121)})$

$$x(t) = t^3 + c_1\,t - 8\,t^2 + c_2 \tag{122}$$

Now equating **(122)** to **(106)**
$\mathbf{(122)} - \mathbf{(113)}$

$$0 = c_1\,t + c_2 - 18\,t - 3 \tag{123}$$

$PDEtools\text{:-}Solve\left(\mathbf{(123)}, \{c_1, c_2\}, independentof = t\right)$

$$\{c_1 = 18, c_2 = 3\} \tag{124}$$

From which the velocity $v_x(t)$ is given by
$subs(\mathbf{(124)}, \mathbf{(120)})$

$$v_x(t) = 3\,t^2 - 16\,t + 18 \tag{125}$$

The equations of motion using tensor notation

Using vector notation to formulate the equations of motion of a particle in Cartesian coordinates is relatively simple. However, for certain problems it may be advantageous to use curvilinear coordinates and / or tensor notation.

restart;
with(*Physics*) : *with*(*Vectors*) :

Cartesian coordinates

Review of Vector notation

Generally speaking, the equations of motion of a particle have the position vector is a function of time, the velocity is its first derivative, and the acceleration is its second derivative. In Cartesian coordinates

$r\_(t) = x(t)\ \_i + y(t)\ \_j + z(t)\ \_k$

$$\vec{r}(t) = x(t)\ \hat{i} + y(t)\ \hat{j} + z(t)\ \hat{k} \qquad \textbf{(126)}$$

diff((**126**), *t*)

$$\dot{\vec{r}}(t) = \dot{x}(t)\ \hat{i} + \dot{y}(t)\ \hat{j} + \dot{z}(t)\ \hat{k} \qquad \textbf{(127)}$$

diff((**127**), *t*)

$$\ddot{\vec{r}}(t) = \ddot{x}(t)\ \hat{i} + \ddot{y}(t)\ \hat{j} + \ddot{z}(t)\ \hat{k} \qquad \textbf{(128)}$$

Newton's 2nd law in an inertial system of reference is given by

$F\_(t) = m \cdot rhs((\mathbf{128}))$

$$\vec{F}(t) = m\left(\ddot{x}(t)\ \hat{i} + \ddot{y}(t)\ \hat{j} + \ddot{z}(t)\ \hat{k}\right) \qquad \textbf{(129)}$$

Tensor notation

Problem

Set a flat 3D spacetime and formulate the equation of movement using tensor notation.

Solution

In Cartesian coordinates, the tensorial form of the equations **(129)** is straightforward. In a flat spacetime - Galilean reference system - the three space coordinates x, y, z form a 3D tensor, as do their first and second derivatives. Set the spacetime to be 3-dimensional and Euclidean, and use *lowercaselatin* indices for the corresponding tensors

Setup(*coordinates* = *cartesian*, *metric* = *Euclidean*, *dimension* = 3, *spacetimeindices* = *lowercaselatin*)

The dimension and signature of the tensor space are set to $[3, (+\ +\ +)]$

Systems of spacetime coordinates are: $\{X = (x, y, z)\}$

The Euclidean metric in coordinates $[x, y, z]$

$$g_{\mu, \nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$[coordinatesystems = \{X\}, dimension = 3, metric = \{(1, 1) = 1, (2, 2) = 1, (3, 3) = 1\},$ $spacetimeindices = lowercaselatin]$ **(130)**

The position, velocity, and acceleration vectors are expressed in tensor notation as

$X[j](t)$

$$X_j(t) \qquad \textbf{(131)}$$

diff((**131**), *t*)

$$\dot{X}_j(t) \qquad \textbf{(132)}$$

diff((**132**), *t*)

$$\ddot{X}_j(t) \quad \textbf{(133)}$$

Set a tensor $F_j(t)$ to represent the three Cartesian components of the force

$Define\left(F[j] = [F_x(t), F_y(t), F_z(t)]\right)$

Defined objects with tensor properties

$$\left\{\gamma_a, F_j, \sigma_a, \partial_a, g_{a,b}, \epsilon_{a,b,c}, X_a\right\} \quad \textbf{(134)}$$

Newton's 2nd law **(129)**, now expressed in tensorial notation, is given by

$F[j] = m \cdot$**(133)**

$$F_j = m\,\ddot{X}_j(t) \quad \textbf{(135)}$$

The three differential equations represented by this tensorial form of **(129)** are, as expected,

$TensorArray($**(135)**$, output = setofequations)$

$$\left\{F_x(t) = m\,\ddot{x}(t), F_y(t) = m\,\ddot{y}(t), F_z(t) = m\,\ddot{z}(t)\right\} \quad \textbf{(136)}$$

Things are straightforward in Cartesian coordinates because the components of the line element $\overrightarrow{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ are exactly the components of the tensor $\mathbf{d}(X_j)$

$TensorArray(d\_(X[j]))$

$$\left[\ \mathbf{d}(x) \quad \mathbf{d}(y) \quad \mathbf{d}(z)\ \right] \quad \textbf{(137)}$$

and so, the *form factors* (see related MaplePrimes post) are all equal to 1.

In the case of no external forces $\vec{F}(t) = 0 = F_j$, the equations of motion, whose solution is the trajectory, can be formulated as the equations of the path of minimal length between two points, a geodesic. Since in this case the spacetime is flat, the geometry is Euclidean, as can be seen in the 3x3 identity metric matrix **(130)**, and so the previously mentioned two points lie on a plane; the geodesic is a straight line. The differential equations of this geodesic are thus the equations of motion **(135)** with $F_j = 0$, and can be computed using Geodesics

$Geodesics(t)$

$$[\ddot{z}(t) = 0, \ddot{y}(t) = 0, \ddot{x}(t) = 0] \quad \textbf{(138)}$$

$Geodesics(t, output = solutions)$

$$\{x(t) = c_1\,t + c_2, y(t) = c_3\,t + c_4, z(t) = c_5\,t + c_6\} \quad \textbf{(139)}$$

This formulation *also works when there are forces*: equate the output to F_j

$F[\sim a] = m \cdot lhs(Geodesics(t, tensornotation))$

$$F^{\,a} = m\,\ddot{X}^{\,a}(t) \quad \textbf{(140)}$$

$TensorArray($**(140)**$)$

$$\left[\ F_x(t) = m\,\ddot{x}(t) \quad F_y(t) = m\,\ddot{y}(t) \quad F_z(t) = m\,\ddot{z}(t)\ \right] \quad \textbf{(141)}$$

So **(140)** is the tensorial form of Newton's 2^{nd} law; as shown in the following problem, it can always be computed in this way using $Geodesics(t, tensornotation)$.

Curvilinear coordinates

Review of Vector notation

In the case of curvilinear coordinates, for example cylindrical or spherical, the form of these equations is obtained by changing the basis and coordinates used to represent the position vector
(126) $\equiv \vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$.

In cylindrical coordinates,
CompactDisplay((rho, *_rho*, phi, *_phi*, *z*, *r_*, *F_*) (*t*))

$$\rho(t) \textit{ will now be displayed as } \rho$$
$$\hat{\rho}(t) \textit{ will now be displayed as } \hat{\rho}$$
$$\phi(t) \textit{ will now be displayed as } \phi$$
$$\hat{\phi}(t) \textit{ will now be displayed as } \hat{\phi}$$
$$z(t) \textit{ will now be displayed as } z$$
$$\vec{r}(t) \textit{ will now be displayed as } \vec{r}$$
$$\vec{F}(t) \textit{ will now be displayed as } \vec{F} \tag{142}$$

*r_*(*t*) = *ChangeBasis*(*rhs*(**(126)**), *cylindrical*, *alsocomponents*)

$$\vec{r} = z\,\hat{k} + \hat{\rho}\,\rho \tag{143}$$

where, since in **(126)** the coordinates (x, y, z) depend on *t*, in **(143)** above not just $\rho(t)$ and $z(t)$ but also the unit vector $\hat{\rho}(t)$ all depend on *t*. After having set a compact display for functions, you can always use the *show* command to see their dependency
show

$$\vec{r}(t) = z(t)\,\hat{k} + \hat{\rho}(t)\,\rho(t) \tag{144}$$

The velocity is computed as usual, by differentiating
diff(**(143)**, *t*)

$$\dot{\vec{r}} = \dot{z}\,\hat{k} + \dot{\phi}\,\hat{\phi}\,\rho + \hat{\rho}\,\dot{\rho} \tag{145}$$

For the acceleration,
diff(**(145)**, *t*)

$$\ddot{\vec{r}} = \hat{\rho}\left(-\dot{\phi}^2\,\rho + \ddot{\rho}\right) + \hat{\phi}\left(\ddot{\phi}\,\rho + 2\,\dot{\phi}\,\dot{\rho}\right) + \ddot{z}\,\hat{k} \tag{146}$$

Newton's 2^{nd} law becomes
*F_*(*t*) = *m*·*rhs*(**(146)**)

$$\vec{F} = m\left(\hat{\rho}\left(-\dot{\phi}^2\,\rho + \ddot{\rho}\right) + \hat{\phi}\left(\ddot{\phi}\,\rho + 2\,\dot{\phi}\,\dot{\rho}\right) + \ddot{z}\,\hat{k}\right) \tag{147}$$

Tensor notation

Problem
Rewrite the equation of movement in cylindrical coordinates using tensor notation.

Solution

We start with the same *setup* of the previous problem:

$Setup(coordinates = cartesian, metric = Euclidean, dimension = 3, spacetimeindices = lowercaselatin)$

Systems of spacetime coordinates are: $\{X = (x, y, z)\}$

The Euclidean metric in coordinates $[x, y, z]$

$$g_{a,b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[coordinatesystems = \{X\}, dimension = 3, metric = \{(1, 1) = 1, (2, 2) = 1, (3, 3) = 1\}, spacetimeindices = lowercaselatin] \quad \textbf{(148)}$$

The tensorial form could be obtained by transforming the tensorial form in Cartesian coordinates, **(140)** $\equiv F^a = \ddot{X}^a(t)$ using the TransformCoordinates command with the transformation from (x, y, z) to (ρ, ϕ, z)

$tr := [X] =\sim ChangeCoordinates([X], cylindrical)$

$$tr := [x = \rho\cos(\phi), y = \rho\sin(\phi), z = z] \quad \textbf{(149)}$$

However, it is simpler and more instructive to only *transform the underlying metric* $g_{a,b}$. This has the advantage that all the geometrical subtleties of curvilinear coordinates, like scale factors and the dependency of unit vectors on curvilinear coordinates, are automatically and succinctly encoded in the metric. To transform and set the result as the new metric all in one go, use the *setmetric* optional argument

$TransformCoordinates(tr, g\_[j, k], [\text{rho}, \text{phi}, z], setmetric):$

Coordinates: $[\rho, \phi, z]$. *Signature:* $(+ + +)$

$$g_{a,b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

---

(150)

The computation of geodesics, that is the equation of movement in the absence of forces, can now be computed directly using Geodesics; it assumes that the coordinates (ρ, ϕ, z) depend on a parameter, the first argument passed:

$Geodesics(t)$

$$\left[\ddot{\rho} = \dot{\phi}^2 \rho, \ddot{\phi} = -\frac{2\dot{\phi}\dot{\rho}}{\rho}, \ddot{z} = 0\right] \quad \textbf{(151)}$$

These are the same equations shown in **(147)** after taking $\vec{F} = 0$. One of the interesting features of computing with tensors is that, as previously mentioned, the geometrical subtleties of curvilinear coordinates are automatically encoded in the metric **(150)**.

To understand how the geodesic equations in **(151)** are computed in one go, one can perform the calculation in steps:

1. Make ρ a function of t directly in the metric
2. Compute - not the final form of the equations **(151)** - but the intermediate form which expresses the geodesic equation using tensor notation, in terms of Christoffel symbols
3. Compute the components of this tensorial equation for the geodesic (using TensorArray)

For step 1, we have
subs(rho = rho(*t*), g_[ ])

$$g_{a,b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(152)}$$

show

$$g_{a,b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho(t)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(153)}$$

Set this metric so $\rho \equiv \rho(t)$
Setup(**(153)**) :

Coordinates: $[\rho, \phi, z]$. *Signature:* $(+\ +\ +)$

$$g_{a,b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

---

(154)

For step 2, the geodesic equations in tensor notation with coordinates depending on the time *t* are computed by passing the optional argument *tensornotation*
Geodesics(*t*, *tensornotation*)

$$\ddot{X}^{a}(t) + \Gamma^{a}{}_{b,c}\,\dot{X}^{b}(t)\,\dot{X}^{c}(t) = 0 \quad \textbf{(155)}$$

Step 3: compute the components of this tensorial equation
TensorArray(**(155)**, *output* = *listofequations*)

$$\left[-\dot{\phi}^2\,\rho + \ddot{\rho} = 0,\ \frac{\ddot{\phi}\,\rho + 2\,\dot{\phi}\,\dot{\rho}}{\rho} = 0,\ \ddot{z} = 0\right] \quad \textbf{(156)}$$

This is the expected result, the same as **(151)**.

Having the tensorial equation **(155)** is also useful for formulating the equations of motion *in the presence of force*. For this purpose, redefine the contravariant tensor F^j to represent the force in the cylindrical basis

$Define\left(F[\sim j] = \left[F_\rho(t), F_\phi(t), F_z(t)\right]\right)$

Defined objects with tensor properties

$$\left\{\nabla_a, \gamma_a, F_j, \sigma_a, R_{a,b}, R_{a,b,c,d}, C_{a,b,c,d}, \partial_a, g_{a,b}, \Gamma_{a,b,c}, G_{a,b}, \epsilon_{a,b,c}, X_a\right\} \tag{157}$$

Now, Newton's 2nd law, expressed using vector notation as **(147)**, in tensor notation is given as done in **(140)**, using $Geodesics(t, tensornotation)$

$F[\sim a] = m \cdot lhs(Geodesics(t, tensornotation))$

$$F^a = m\left(\ddot{X}^a(t) + \Gamma^a{}_{b,c}\,\dot{X}^b(t)\,\dot{X}^c(t)\right) \tag{158}$$

$TensorArray(\mathbf{(158)})$

$$\left[\begin{array}{ccc} F_\rho(t) = m\left(-\dot{\phi}^2\rho + \ddot{\rho}\right) & F_\phi(t) = \dfrac{m\left(\ddot{\phi}\rho + 2\dot{\phi}\dot{\rho}\right)}{\rho} & F_z(t) = m\ddot{z} \end{array}\right] \tag{159}$$

So **(158)** is the tensorial form of Newton's 2^{nd} law, where we recall (see related MaplePrimes post) that, to obtain the *vector* components of $\vec{F}(t)$ from these *tensor* components F^a, we need to multiply the latter by the scale factors $(1, \rho, 1)$. Therefore, the component of $\vec{F}$ in the direction of $\hat{\phi}$ is equal to

$\vec{F}(t)\,.\,\hat{\phi} = \rho \cdot F_\phi(t) = \rho \cdot m\left(\ddot{\phi} + \dfrac{2\dot{\rho}\dot{\phi}}{\rho}\right)$.

Many-particle systems

Center of Mass

Given a system of n particles of masses m_i with positions $\vec{r}_i$ in some frame of reference K, the center of mass of the system is defined as

$$\vec{R} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$$

The velocity of the center of mass is thus

$$\vec{V} = \dot{\vec{R}} = \frac{\sum_{i=1}^{n} m_i \dot{\vec{r}}_i}{\sum_{i=1}^{n} m_i}$$

Problem

Consider a system of particles viewed from two systems of reference, K and K', that move with respect to each other at a constant velocity $\vec{V}$ measured in K. Show that:

a) When $\vec{V}$ is the velocity of the center of mass, the total momentum $\vec{P}\,'$ measured in K' is equal to 0.

b) The relation between $\vec{P}$ and the velocity $\vec{V}$ of the center of mass, both measured in K, is the same as the relation $\vec{p} = m\,\vec{v}$ between the momentum, velocity and mass of a single particle of mass $\mu = \sum_{i=1}^{n} m_i$.

Solution

restart;
with(*Physics*:-*Vectors*) :

The relation between the velocities of each particle in the systems K and K' *are*
v_[i] = V_ + v'_[i]

$$\vec{v}_i = \vec{V} + \vec{v'}_i \tag{160}$$

The momenta of a single particle in K and K' are thus related by
$m[i]\cdot$**(160)**

$$m_i\,\vec{v}_i = m_i\left(\vec{V} + \vec{v'}_i\right) \tag{161}$$

and the total momentum by
map(*Sum*, **(161)**, $i = 1..n$)

$$\sum_{i=1}^{n} m_i\,\vec{v}_i = \sum_{i=1}^{n} m_i\left(\vec{V} + \vec{v'}_i\right) \tag{162}$$

expand(**(162)**)

$$\sum_{i=1}^{n} m_i\,\vec{v}_i = \vec{V}\left(\sum_{i=1}^{n} m_i\right) + \left(\sum_{i=1}^{n} m_i\,\vec{v'}_i\right) \tag{163}$$

a) Taking $\vec{V}$ as the velocity of the center of mass,

$$subs\left(\vec{V} = \frac{\sum_{i=1}^{n} m_i\,\vec{v}_i}{\sum_{i=1}^{n} m_i}, \mathbf{(163)}\right)$$

$$\sum_{i=1}^{n} m_i\,\vec{v}_i = \left(\sum_{i=1}^{n} m_i\,\vec{v}_i\right) + \left(\sum_{i=1}^{n} m_i\,\vec{v'}_i\right) \tag{164}$$

where the left-hand side is the total momentum $\sum_{i=1}^{n} m_i\,\vec{v}_i = \vec{P}$ measured in the K system, and on the right-hand side $\vec{P}\,' = \sum_{i=1}^{n} m_i\,\vec{v'}_i$ is the total momentum measured in the K' system. Passing all terms to the left, we have
(*rhs* − *lhs*)(**(164)**) = 0

$$\sum_{i=1}^{n} m_i\,\vec{v'}_i = 0 \tag{165}$$

that is, when in **(163)** $\vec{V} = \frac{\sum_{i=1}^{n} m_i \vec{v}_i}{\sum_{i=1}^{n} m_i}$ it follows that $\vec{P}' = \sum_{i=1}^{n} m_i \vec{v'}_i = 0$.

---

b) Inserting **(165)** $\equiv \vec{P}' = 0$ into **(163)**, we get

subs$\left(\mathbf{(165)}, \sum_{i=1}^{n} m_i \vec{v}_i = P\_, \mathbf{(163)}\right)$

$$\vec{P} = \vec{V}\left(\sum_{i=1}^{n} m_i\right) \quad \mathbf{(166)}$$

therefore the relation between $\vec{P}$ and the velocity $\vec{V}$ of the center of mass is the same as the relation $\vec{p} = m\,\vec{v}$ between the momentum, velocity and mass of a single particle of mass $\mu = \sum_{i=1}^{n} m_i$.

The equations of motion

Problem

Show that, for a system of particles with total mass $\mu = \sum_{i=1}^{n} m_i$, Newton's 2nd law for each particle $\vec{F}_i = m_i \ddot{\vec{r}}_i$ implies that $\vec{F}_{ext} = \mu\,\ddot{\vec{R}}$, where $\vec{R}$ is the center of mass and $\vec{F}_{ext}$ is the *external* force applied to the system (it excludes the force that the particles exercise on each other).

Solution

restart;
with(*Physics*:-*Vectors*) :
CompactDisplay((*r*_[*i*], *R*_)(*t*))

$$\vec{R}(t) \text{ will now be displayed as } \vec{R}$$
$$\vec{r}(t) \text{ will now be displayed as } \vec{r} \quad \mathbf{(167)}$$

The force $\vec{F}_i$ acting on the i^{th} particle has two parts

Sum(*f*_[*i*, *j*], *j* = 1 ..*n*) + *f*_[*i*, *ext*]

$$\left(\sum_{j=1}^{n} \vec{f}_{i,j}\right) + \vec{f}_{i,\,ext} \quad \mathbf{(168)}$$

where the first term represents the forces acting on the i^{th} particle from the other particles, and $\vec{f}_{i,\,ext}$ represents all other forces acting on the i^{th} particle. The total force acting on the system is thus the sum of these forces from 1 to n

$F\_ = Sum((\mathbf{168}), i = 1..n)$

$$\vec{F} = \sum_{i=1}^{n} \left(\left(\sum_{j=1}^{n} \vec{f}_{i,j} \right) + \vec{f}_{i,\, ext} \right) \tag{169}$$

$expand((\mathbf{169}))$

$$\vec{F} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \vec{f}_{i,j} \right) + \left(\sum_{i=1}^{n} \vec{f}_{i,\, ext} \right) \tag{170}$$

The first term on the right-hand side is zero: due to Newton's 3^{rd} law, to each $\vec{f}_{i,j}$ term in that sum corresponds another term $\vec{f}_{j,i} = -\vec{f}_{i,j}$,

$expand(subs(f\_[i,j] = 0, (\mathbf{170})))$

$$\vec{F} = \sum_{i=1}^{n} \vec{f}_{i,\, ext} \tag{171}$$

In turn, $\vec{F}$ on the left-hand side can be written as the sum of $m\, \ddot{\vec{r}}_i$ over the n particles

$F\_ = Sum(m[i] \cdot diff(r\_[i](t), t, t), i = 1..n)$

$$\vec{F} = \sum_{i=1}^{n} m_i\, \ddot{\vec{r}}_i \tag{172}$$

$subs((\mathbf{172}), (\mathbf{171}))$

$$\sum_{i=1}^{n} m_i\, \ddot{\vec{r}}_i = \sum_{i=1}^{n} \vec{f}_{i,\, ext} \tag{173}$$

Introducing the center of mass

$R\_(t) = Sum(m[i]\, r\_[i](t), i = 1 .. n) / Sum(m[i], i = 1 .. n)$

$$\vec{R} = \frac{\sum_{i=1}^{n} m_i\, \vec{r}_i}{\sum_{i=1}^{n} m_i} \tag{174}$$

$diff((\mathbf{174}), t, t)$

$$\ddot{\vec{R}} = \frac{\sum_{i=1}^{n} m_i\, \ddot{\vec{r}}_i}{\sum_{i=1}^{n} m_i} \tag{175}$$

Substituting the expression **(173)**, including $\sum_{i=1}^{n} m_i = \mu$ and multiplying by μ

$\text{mu} \cdot subs\left((\mathbf{173}), \sum_{i=1}^{n} m_i = M, (\mathbf{175})\right)$

$$\mu \ddot{\vec{R}} = \frac{\mu \left(\sum_{i=1}^{n} \vec{f}_{i, ext} \right)}{M} \tag{176}$$

Renaming $\sum_{i=1}^{n} \vec{f}_{i, ext} = \vec{F}_{ext}$ as the total external force acting upon the system,

subs(*rhs*((**176**)) = *F*_[*ext*], (**176**))

$$\mu \ddot{\vec{R}} = \vec{F}_{ext} \tag{177}$$

which is the expected result.

Problem

Show that :

a) The total linear momentum $\vec{P}$ satisfies $\dot{\vec{P}} = \vec{F}_{ext}$

b) The total torque $\vec{N} = \dot{\vec{L}}$ satisfies $\vec{N} = \sum_{i=1}^{n} \vec{r}_i \times \vec{f}_{i, ext}$

Solution

restart;
with(*Physics*:-*Vectors*) :
CompactDisplay((*r*_, *l*_, *p*_, *R*_, *L*_, *N*_, *P*_)(*t*))

$$\begin{gathered}\vec{r}(t) \text{ will now be displayed as } \vec{r}\\ \vec{l}(t) \text{ will now be displayed as } \vec{l}\\ \vec{p}(t) \text{ will now be displayed as } \vec{p}\\ \vec{R}(t) \text{ will now be displayed as } \vec{R}\\ \vec{L}(t) \text{ will now be displayed as } \vec{L}\\ \vec{N}(t) \text{ will now be displayed as } \vec{N}\\ \vec{P}(t) \text{ will now be displayed as } \vec{P}\end{gathered} \tag{178}$$

a) The total linear momentum is defined as $\vec{P} = \sum_{i=1}^{n} \vec{p}_i$. Starting from **(173)** derived in the previous problem

(173)

$$\sum_{i=1}^{n} m_i \ddot{\vec{r}}_i = \sum_{i=1}^{n} \vec{f}_{i, ext} \tag{179}$$

The summand can be written as $m_i \left(\ddot{\vec{r}}_i \right) = m_i \dot{\vec{v}}_i(t) = \dot{\vec{p}}_i(t)$

$$subs\left(diff(r\_[i](t), t, t) = \frac{diff(p\_[i](t), t)}{m[i]}, \mathbf{(173)} \right)$$

$$\sum_{i=1}^{n} \dot{\vec{p}}_i = \sum_{i=1}^{n} \vec{f}_{i,\,ext} \tag{180}$$

Factoring out $\frac{d}{dt}$ gives

$subs\left(\sum_{i=1}^{n} diff\left(p\_{i}(t), t\right) = \%diff\left(\left(\sum_{i=1}^{n} p\_{i}(t)\right), t\right), \mathbf{(180)}\right)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} \vec{f}_{i,\,ext} \tag{181}$$

$subs\left(\sum_{i=1}^{n} \vec{p}_i(t) = P\_(t), \sum_{i=1}^{n} \vec{f}_{i,\,ext} = F\_[ext], \mathbf{(181)}\right)$

$$\frac{\mathrm{d}\vec{P}}{\mathrm{d}t} = \vec{F}_{ext} \tag{182}$$

---

b) From **(68)**, the angular momentum of the i^{th} particle is given by

$subs\left(L\_ = l\_[i], r\_ = r\_[i], p\_ = p\_[i], \mathbf{(65)}\right)$

$$\vec{l}_i = \vec{r}_i \times \vec{p}_i \tag{183}$$

Differentiating, the torque of the i^{th} particle $\vec{n}_i = \dot{\vec{l}}_i$ is

$diff\left(\mathbf{(183)}, t\right)$

$$\dot{\vec{l}}_i = \dot{\vec{r}}_i \times \vec{p}_i + \vec{r}_i \times \dot{\vec{p}}_i \tag{184}$$

The first term on the right-hand side is zero since $\dot{\vec{r}}_i$ and $\vec{p}_i$ are parallel.

$eval\left(\mathbf{(184)}, diff\left(r\_[i](t), t\right) = 0\right)$

$$\dot{\vec{l}}_i = \vec{r}_i \times \dot{\vec{p}}_i \tag{185}$$

In turn, the force $\vec{F}_i$ acting on the i^{th} particle is given by **(168)**

$F\_[i] = \mathbf{(168)}$

$$\vec{F}_i = \left(\sum_{j=1}^{n} \vec{f}_{i,j}\right) + \vec{f}_{i,\,ext} \tag{186}$$

Introducing $\dot{\vec{p}}_i = \vec{F}_i$, and the expression of $\vec{F}_i$ into the expression for $\dot{\vec{l}}_i$

$subs\left(diff\left(p\_[i](t), t\right) = F\_[i], \mathbf{(186)}, \mathbf{(185)}\right)$

$$\dot{\vec{l}}_i = \vec{r}_i \times \left(\left(\sum_{j=1}^{n} \vec{f}_{i,j}\right) + \vec{f}_{i,\,ext}\right) \tag{187}$$

The same way the total angular momentum is defined as $\vec{L} = \sum_{i=1}^{n} \vec{l}_i$, the total torque $\vec{N} = \sum_{i=1}^{n} \vec{n}_i$ is the sum of this expression over n particles

$map(Sum, subs(diff(l\_[i](t), t) = n\_[i], \mathbf{(187)}), i = 1..n)$

$$\sum_{i=1}^{n} \vec{n}_i = \sum_{i=1}^{n} \vec{r}_i \times \left(\left(\sum_{j=1}^{n} \vec{f}_{i,j} \right) + \vec{f}_{i,\,ext} \right) \qquad \mathbf{(188)}$$

$expand(\mathbf{(188)})$

$$\sum_{i=1}^{n} \vec{n}_i = \left(\sum_{i=1}^{n} \vec{r}_i \times \left(\sum_{j=1}^{n} \vec{f}_{i,j} \right) \right) + \left(\sum_{i=1}^{n} \vec{r}_i \times \vec{f}_{i,\,ext} \right) \qquad \mathbf{(189)}$$

The left-hand side is, by definition, the total torque $\sum_{i=1}^{n} \vec{n}_i = \vec{N}$. On the right-hand side, the first sum is equal to 0 due to Newton's 3^{rd} law: for each term with $\vec{f}_{i,j}$ in the the double sum ($i \neq j$) there is another term with $\vec{f}_{j,i} = -\vec{f}_{i,j}$

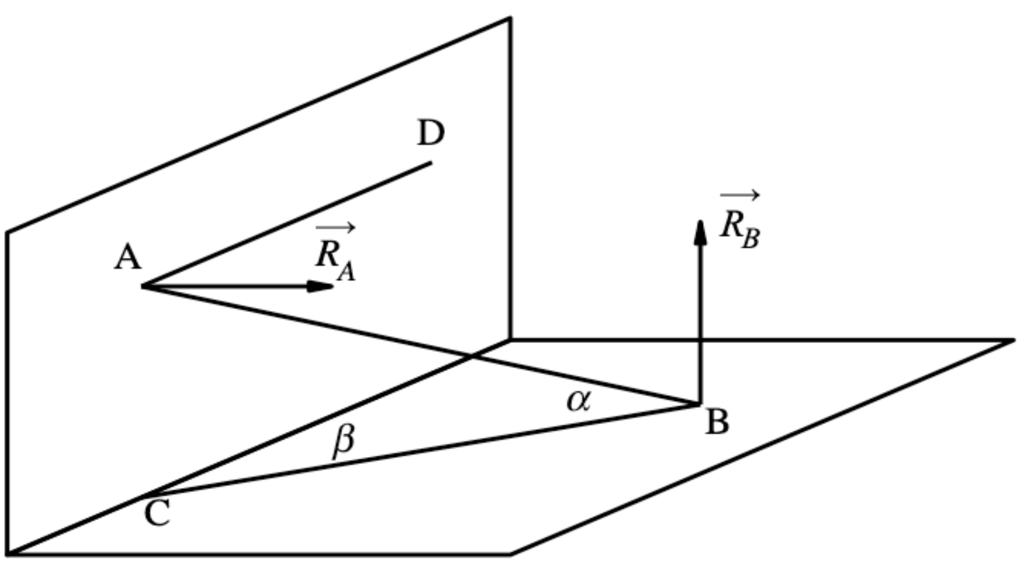
$N\_ = expand(subs(f\_[i, j] = 0, rhs(\mathbf{(189)})))$

$$\vec{N} = \sum_{i=1}^{n} \vec{r}_i \times \vec{f}_{i,\,ext} \qquad \mathbf{(190)}$$

Static: reactions of planes and tensions on cables

Problem

A bar **AB** of weight **w** and length **L** has one extreme on a horizontal plane and the other on a vertical place, and is kept in that position by two cables **AD** and **BC**. The bar forms an angle α with the horizontal plane and its projection **BC** over this plane forms an angle β with the vertical plane. The cable **BC** is on the same vertical plane as the bar.



Determine the reactions of the planes at **A** and **B** as well as the tensions on the cables.

Solution

There are two equations that contain information about the state of equilibrium of a system. The first one states that the center of mass of the body is not accelerated as long as the sum of the forces acting on the body are equal to zero. The second one states that the rotation of the body around its center of masses is unchanging (if it is not rotating then it stays that way) as long as the sum of the moments of the forces acting on the body (that is, the total torque) is zero. These two equations involve the reactions of the planes and the tensions on the cables, so from them we can obtain the solution to the overall problem. There is no friction so it is also clear that the reactions $\vec{R}_A$ and $\vec{R}_B$ are perpendicular to the planes, as shown in the figure, and the tensions $\vec{T}_A$ and $\vec{T}_B$ on the cables have direction **AD** and **BC**, respectively.

The steps to solving this problem are:

1. Determine each force $\vec{F}$ acting on the bar as well as its application point $\vec{r}$.
2. Equate the sum of the forces $\vec{F}$ to zero.
3. Equate the sum of the moments $\vec{r} \times \vec{F}$ to zero.
4. Solve these two vectorial equations for $\vec{R}_A$, $\vec{R}_B$, $\vec{T}_A$, and $\vec{T}_B$, representing the reactions of the planes at the points of contact **A** and **B** and the tensions of the cables attached to the bar at **A** and **B**, respectively.

restart :
with(*Physics*:-*Vectors*) :

The forces acting on the bar are its weight $\vec{w}$ and the reactions and tensions $\vec{R}_A$, $\vec{R}_B$, $\vec{T}_A$, and $\vec{T}_B$. So the two equilibrium equations are

$eq[1] := w\_ + R\_[A] + R\_[B] + T\_[A] + T\_[B] = 0$

$$eq_1 := \vec{w} + \vec{R}_A + \vec{R}_B + \vec{T}_A + \vec{T}_B = 0 \quad \textbf{(191)}$$

$eq[2] := r\_[w] \times w\_ + r\_[A] \times R\_[A] + r\_[B] \times R\_[B] + r\_[A] \times T\_[A] + r\_[B] \times T\_[B] = 0$

$$eq_2 := \vec{r}_w \times \vec{w} + \vec{r}_A \times \vec{R}_A + \vec{r}_B \times \vec{R}_B + \vec{r}_A \times \vec{T}_A + \vec{r}_B \times \vec{T}_B = 0 \quad \textbf{(192)}$$

where, in the input above, to enter the cross products you can use *&x* or the × operator from the palette of Common Symbols. Set the origin and orientation of the reference system to project these vectors; any choice will do, but a good one will simplify the algebraic manipulations. We will set the origin at the point **B**, with the vertical z axis in the direction of the reaction $\vec{R}_B$ such that $\vec{r}_B = 0$, the y axis in the direction of $\vec{R}_A$, and the x axis in the remaining direction, anti-parallel to $\vec{T}_A$. With these choices, the vectors entering eq_1 and eq_2 are projected as follows

$R\_[B] := \text{abs}(R[B])\_k$

$$\vec{R}_B := |R_B|\,\hat{k} \quad \textbf{(193)}$$

where $|R_B|$ represents the norm of $\vec{R}_B$, to be determined

$r\_[B] := 0$

$$\vec{r}_B := 0 \quad \textbf{(194)}$$

$R\_[A] := \text{abs}(R[A])\_j$

$$\vec{R}_A := |R_A|\,\hat{j} \tag{195}$$

and where $|R_A|$ is also to be determined. This reaction $\vec{R}_A$ is applied to the bar at **A**, represented by $\vec{r}_A$; its component along the *x* axis is obtained by projecting the segment **BA** onto the horizontal plane ($L\cos(\alpha)$), resulting in **BC**, and then onto the intersection of the two planes

r_[A] := L cos(alpha)·(cos(beta) _i + sin(2 Pi − beta) _j) + L sin(alpha)_k

$$\vec{r}_A := -L\cos(\alpha)\sin(\beta)\,\hat{j} + L\cos(\alpha)\cos(\beta)\,\hat{i} + L\sin(\alpha)\,\hat{k} \tag{196}$$

For the other vectors we have

T_[A] := − abs(T[A])_i

$$\vec{T}_A := -|T_A|\,\hat{i} \tag{197}$$

T_[B] := abs(T[B]) cos(beta) _i + abs(T[B]) sin(2 Pi − beta)_j

$$\vec{T}_B := |T_B|\cos(\beta)\,\hat{i} - |T_B|\sin(\beta)\,\hat{j} \tag{198}$$

where $|T_A|$ and $|T_B|$ are to be determined

w_ := −abs(w) _k

$$\vec{w} := -|w|\,\hat{k} \tag{199}$$

$$r\_[w] := \frac{r\_[A]}{2}$$

$$\vec{r}_w := -\frac{L\cos(\alpha)\sin(\beta)\,\hat{j}}{2} + \frac{L\cos(\alpha)\cos(\beta)\,\hat{i}}{2} + \frac{L\sin(\alpha)\,\hat{k}}{2} \tag{200}$$

The two equilibrium equations now appear as

eq[1]

$$-|w|\,\hat{k} + |R_A|\,\hat{j} + |R_B|\,\hat{k} - |T_A|\,\hat{i} + |T_B|\cos(\beta)\,\hat{i} - |T_B|\sin(\beta)\,\hat{j} = 0 \tag{201}$$

eq[2]

$$\frac{L\cos(\alpha)\sin(\beta)\,|w|\,\hat{i}}{2} + \frac{L\cos(\alpha)\cos(\beta)\,|w|\,\hat{j}}{2} - L\sin(\alpha)\,|R_A|\,\hat{i} + L\cos(\alpha)\cos(\beta)\,|R_A|\,\hat{k} - L\sin(\alpha)\,|T_A|\,\hat{j} - L\cos(\alpha)\sin(\beta)\,|T_A|\,\hat{k} = 0 \tag{202}$$

These two vectorial equations represent a system of six equations which can be obtained by equating each of the coefficients of $\hat{i}$, $\hat{j}$, and $\hat{k}$ in each of the equations to zero; that is, taking the components of the vectorial equations along each axis

Eq[1, 2, 3] := *seq*(*Component*(*lhs*(*eq*[1]), *n*) = 0, *n* = 1 ..3)

$$Eq_{1,2,3} := -|T_A| + |T_B|\cos(\beta) = 0,\ |R_A| - |T_B|\sin(\beta) = 0,\ -|w| + |R_B| = 0 \tag{203}$$

Eq[4, 5, 6] := *seq*(*Component*(*lhs*(*eq*[2]), *n*) = 0, *n* = 1 ..3)

$$Eq_{4,5,6} := \frac{L\cos(\alpha)\sin(\beta)\,|w|}{2} - L\sin(\alpha)\,|R_A| = 0,\ \frac{L\cos(\alpha)\cos(\beta)\,|w|}{2} - L\sin(\alpha)\,|T_A| = 0,\ L\cos(\alpha)\cos(\beta)\,|R_A| - L\cos(\alpha)\sin(\beta)\,|T_A| = 0 \tag{204}$$

So the system of equations to be solved is

sys := {*Eq*[1, 2, 3], *Eq*[4, 5, 6]}

$$sys := \left\{ \frac{L\cos(\alpha)\cos(\beta)\,|w|}{2} - L\sin(\alpha)\,|T_A| = 0, L\cos(\alpha)\cos(\beta)\,|R_A| - L\cos(\alpha)\sin(\beta)\,|T_A| = 0, \frac{L\cos(\alpha)\sin(\beta)\,|w|}{2} - L\sin(\alpha)\,|R_A| = 0, -|w| + |R_B| = 0, |R_A| - |T_B|\sin(\beta) = 0, -|T_A| + |T_B|\cos(\beta) = 0 \right\} \quad \textbf{(205)}$$

The unknowns are

$var := \{\mathrm{abs}(R[A]), \mathrm{abs}(R[B]), \mathrm{abs}(T[A]), \mathrm{abs}(T[B])\}$

$$var := \{|R_A|, |R_B|, |T_A|, |T_B|\} \quad \textbf{(206)}$$

and the solution is

$solve(sys, var)$

$$\left\{|R_A| = \frac{\cos(\alpha)\,|w|\sin(\beta)}{2\sin(\alpha)}, |R_B| = |w|, |T_A| = \frac{\cos(\alpha)\,|w|\cos(\beta)}{2\sin(\alpha)}, |T_B| = \frac{\cos(\alpha)\,|w|}{2\sin(\alpha)}\right\} \quad \textbf{(207)}$$

Lagrange equations

$restart;$
$with(Physics\text{:-}Vectors):$
$CompactDisplay((r\_, v\_)(t))$

$$\vec{r}(t) \textit{ will now be displayed as } \vec{r}$$
$$\vec{v}(t) \textit{ will now be displayed as } \vec{v} \quad \textbf{(208)}$$

In the case of a closed system, or a system in a constant external field, the equations of motions can also be derived from the knowledge of the kinetic energy T and the potential energy U. For this purpose, construct the Lagrange function $L = T - U$ and derive the equations of motion as the Lagrange equations for L.

For closed systems, the potential energy $U(\vec{r}_i)$ is related to the force acting on each particle by the equation $\vec{F}_i = -\nabla_i(U(\vec{r}_1, ..., \vec{r}_n))$. Formally, $\nabla_i \equiv \frac{\partial}{\partial \vec{r}_i}$ is the gradient operator in the basis onto which $\vec{r}_i$ is projected, and with respect to its coordinates - say in Cartesian basis and coordinates $\nabla_i = \hat{i}\cdot\left(\frac{\partial}{\partial x_i}\right) + \hat{j}\cdot\left(\frac{\partial}{\partial y_i}\right) + \hat{k}\cdot\left(\frac{\partial}{\partial z_i}\right)$.

The kinetic energy - say T - of a single particle is given by

$T := \frac{m \cdot v\_(t)^2}{2}$

$$T := \frac{m\,\vec{v}^{\,2}}{2} \quad \textbf{(209)}$$

Since the kinetic energy T is additive, a system of n particles has

$T := \%sum\left(\frac{m \cdot v\_[i](t)^2}{2}, i = 1..n\right)$

$$T := \sum_{i=1}^{n} \frac{m\,\vec{v}_i^{\,2}}{2} \tag{210}$$

where $\vec{v}_i$ is the velocity of the i^{th} particle. Generally speaking, the potential energy $U(\vec{r}_1, ..., \vec{r}_n)$ of the system is a function of the coordinates $\vec{r}_i$ of the n particles, and the Lagrangian is defined as

$L = T - U(\vec{r}_1, ..., \vec{r}_n)$

$$L = \left(\sum_{i=1}^{n} \frac{m\,\vec{v}_i^{\,2}}{2}\right) - U(\vec{r}_1, ..., \vec{r}_n) \tag{211}$$

The potential energy $U(\vec{r}_i)$ is related to the force acting on each particle by the equation $\vec{F}_i = -\nabla_i\left(U(\vec{r}_1, ..., \vec{r}_n)\right)$. Formally, $\nabla_i \equiv \frac{\partial}{\partial \vec{r}_i}$ is the gradient operator in the basis onto which $\vec{r}_i$ is projected. Knowing the Lagrangian, we can derive the (Lagrange) equations of motion as

$\%diff(\%diff(L, v\_[i]), t) = \%diff(L, r\_[i])$

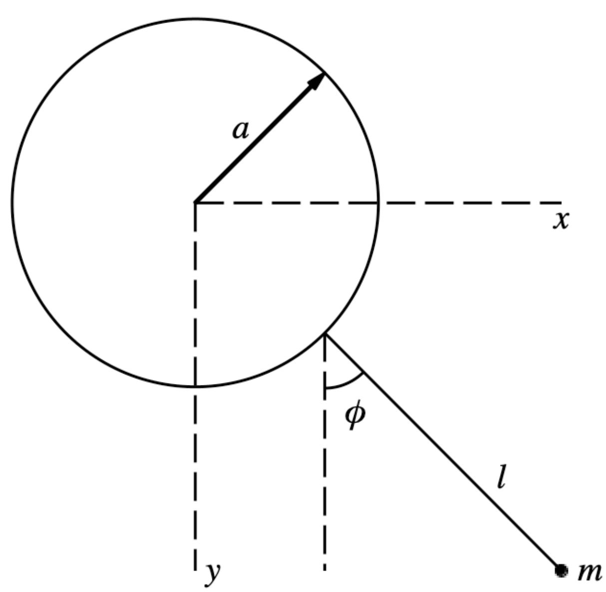
$$\frac{\partial}{\partial t} \frac{\mathrm{d}L}{\mathrm{d}\vec{v}_i} = \frac{\mathrm{d}L}{\mathrm{d}\vec{r}_i} \tag{212}$$

Motion of a pendulum

Problem

Determine the Lagrangian and equation of motion of a plane pendulum with a mass m at its extremity and a suspension point which:

a) Moves uniformly over a vertical circumference with a constant frequency ω.



b) Oscillates horizontally on the plane of the pendulum according to $x = \cos(\omega t)$.

c) Is fixed ($\omega = 0$). Integrate the equation of motion for small oscillations.

Solution

restart :
with(*Physics*:-*Vectors*) :
with(*Physics*, *LagrangeEquations*) :

a) The Lagrangian is defined as

$L := T - U$

$$L := T - U \tag{213}$$

where T and U are the kinetic and potential energy of the system, which in this case is constituted by a single point of mass m. The potential energy U is the gravitational energy

$U := -m\, g\, y$

$$U := -m\, g\, y \tag{214}$$

where g is the gravitational constant and the y axis is chosen to be along the vertical, pointing downwards, resulting in the gravitational force $\vec{F}_g = mg\,\hat{j}$. The kinetic energy is

$T := \frac{1}{2}\, m \cdot (v\_ \,.\, v\_)$

$$T := \frac{m\, \|\vec{v}\|^2}{2} \tag{215}$$

To compute this velocity, the position vector $\vec{r}$ of the suspension point of the pendulum,

$r\_ := x\_i + y\_j$

$$\vec{r} := x\,\hat{i} + y\,\hat{j} \tag{216}$$

must be determined. Choosing the x axis to be along the horizontal and the origin of the reference system at the center of the circle (see figure above), the x and y coordinates are given by

$parametric\_equations := [x = a\cos(\omega t) + l\sin(\phi(t)), y = -a\sin(\omega t) + l\cos(\phi(t))]$

$$parametric\_equations := [x = a\cos(\omega t) + l\sin(\phi(t)), y = -a\sin(\omega t) + l\cos(\phi(t))] \tag{217}$$

$CompactDisplay(parametric\_equations)$

$$\phi(t) \textit{ will now be displayed as } \phi \tag{218}$$

$r\_ := eval(r\_, parametric\_equations)$

$$\vec{r} := (a\cos(\omega t) + l\sin(\phi))\,\hat{i} + (-a\sin(\omega t) + l\cos(\phi))\,\hat{j} \tag{219}$$

$v\_ := diff(r\_, t)$

$$\vec{v} := (-a\,\omega\sin(\omega t) + l\,\dot{\phi}\cos(\phi))\,\hat{i} + (-a\,\omega\cos(\omega t) - l\,\dot{\phi}\sin(\phi))\,\hat{j} \tag{220}$$

T

$$\frac{m\left((-a\,\omega\sin(\omega t) + l\,\dot{\phi}\cos(\phi))^2 + (-a\,\omega\cos(\omega t) - l\,\dot{\phi}\sin(\phi))^2\right)}{2} \tag{221}$$

This expression contains products of trigonometric functions, so it can be simplified by combining these products.

$combine(T, trig)$

$$\frac{m\left(\dot{\phi}^2 l^2 - 2\,\dot{\phi}\,a\,l\,\omega\sin(\omega t-\phi) + a^2\omega^2\right)}{2} \tag{222}$$

For the gravitational energy, expressed in terms of the parametric equations of the point of mass m, we have

$U := eval(U, parametric\_equations)$

$$U := -m\,g\,\left(-a\sin(\omega t) + l\cos(\phi)\right) \tag{223}$$

So the requested Lagrangian is

$L := combine(L, trig)$

$$L := -\sin(\omega t-\phi)\,\dot{\phi}\,a\,l\,m\,\omega + \frac{\dot{\phi}^2 l^2 m}{2} + \frac{a^2 m\,\omega^2}{2} + \cos(\phi)\,g\,l\,m - \sin(\omega t)\,a\,g\,m \tag{224}$$

Taking into account that the Lagrangian of a system is defined up to a total derivative with respect to t, we can eliminate the two terms $\frac{m\,a^2\omega^2}{2}$ and $m\,g\,a\sin(\omega t)$ that can be rewritten as total derivatives

$L := subs\left(\omega^2 = 0, \sin(\omega t) = 0, L\right)$

$$L := -\sin(\omega t-\phi)\,\dot{\phi}\,a\,l\,m\,\omega + \frac{\dot{\phi}^2 l^2 m}{2} + \cos(\phi)\,g\,l\,m \tag{225}$$

The equation of motion is Lagrange's equation for ϕ

$LagrangeEquations(L, \phi)$

$$l\,m\left(\cos(\omega t-\phi)\,a\,\omega^2 - \ddot{\phi}\,l - \sin(\phi)\,g\right) = 0 \tag{226}$$

---

b) The steps are the same as in part **a)** but for the expression of the y coordinate, which for this part **b)** is

$parametric\_equations[2] := y = l\cos(\phi(t))$

$$parametric\_equations := [x = a\cos(\omega t) + l\sin(\phi), y = l\cos(\phi)] \tag{227}$$

From which

$r\_ := eval(x\_i + y\_j, parametric\_equations)$

$$\vec{r} := \left(a\cos(\omega t) + l\sin(\phi)\right)\hat{i} + l\cos(\phi)\,\hat{j} \tag{228}$$

$v\_ := diff(r\_, t)$

$$\vec{v} := \left(-a\,\omega\sin(\omega t) + l\,\dot{\phi}\cos(\phi)\right)\hat{i} - \dot{\phi}\sin(\phi)\,\hat{j}\,l \tag{229}$$

$T := \frac{1}{2}\,m\,(v\_ \,.\, v\_)$

$$T := \frac{m\left(\left(-a\,\omega\sin(\omega t) + l\,\dot{\phi}\cos(\phi)\right)^2 + \sin(\phi)^2\,\dot{\phi}^2 l^2\right)}{2} \tag{230}$$

$U := eval(-m\,g\,y, parametric\_equations)$

$$U := -\cos(\phi)\,g\,l\,m \tag{231}$$

The Lagrangian is

$L := combine(T - U, trig)$

$$L := -\frac{\sin(\omega t+\phi)\,\dot{\phi}\,a\,l\,m\,\omega}{2} - \frac{\sin(\omega t-\phi)\,\dot{\phi}\,a\,l\,m\,\omega}{2} - \frac{\cos(2\,\omega t)\,a^2\,m\,\omega^2}{4} + \frac{\dot{\phi}^2\,l^2\,m}{2} + \frac{a^2\,m\,\omega^2}{4} + \cos(\phi)\,g\,l\,m \quad (232)$$

Discarding total derivatives,

$L := subs\left(\omega^2 = 0, \cos(2\,\omega t) = 0, L\right)$

$$L := -\frac{\sin(\omega t+\phi)\,\dot{\phi}\,a\,l\,m\,\omega}{2} - \frac{\sin(\omega t-\phi)\,\dot{\phi}\,a\,l\,m\,\omega}{2} + \frac{\dot{\phi}^2\,l^2\,m}{2} + \cos(\phi)\,g\,l\,m \quad (233)$$

The equation of motion is

$LagrangeEquations(L, \phi)$

$$\frac{l\,m\left(\cos(\omega t+\phi)\,a\,\omega^2 + \cos(\omega t-\phi)\,a\,\omega^2 - 2\,\ddot{\phi}\,l - 2\sin(\phi)\,g\right)}{2} = 0 \quad (234)$$

---

c) Taking $\omega = 0$ in the parametric equations of the problems **a)** or **b)**, we get

$parametric\_equations := eval(parametric\_equations, \omega = 0)$

$$parametric\_equations := [x = a + l\sin(\phi), y = l\cos(\phi)] \quad (235)$$

For simplicity, and without loss of generality, translate the origin of the system of references by a in the x direction

$parametric\_equations := eval(parametric\_equations, a = 0)$

$$parametric\_equations := [x = l\sin(\phi), y = l\cos(\phi)] \quad (236)$$

The next steps are the same as before

$r\_ := eval(x\,\_i + y\,\_j, parametric\_equations)$

$$\vec{r} := l\sin(\phi)\,\hat{i} + l\cos(\phi)\,\hat{j} \quad (237)$$

$v\_ := diff(r\_, t)$

$$\vec{v} := \dot{\phi}\cos(\phi)\,\hat{i}\,l - \dot{\phi}\sin(\phi)\,\hat{j}\,l \quad (238)$$

$T := \frac{1}{2}\,m\,(v\_ \cdot v\_)$

$$T := \frac{m\left(\dot{\phi}^2\cos(\phi)^2\,l^2 + \sin(\phi)^2\,\dot{\phi}^2\,l^2\right)}{2} \quad (239)$$

$T := simplify(T)$

$$T := \frac{\dot{\phi}^2\,l^2\,m}{2} \quad (240)$$

$L := T - U$

$$L := \frac{\dot{\phi}^2\,l^2\,m}{2} + \cos(\phi)\,g\,l\,m \quad (241)$$

The equation of motion is

$LagrangeEquations(L, \phi)$

$$\ddot{\phi}\, l^2\, m + \sin(\phi)\, g\, l\, m = 0 \quad \textbf{(242)}$$

When the oscillation angle ϕ is small, expanding in series up to first order in ϕ

$series(\sin(\phi), \phi, 2)$

$$\phi + O(\phi^3) \quad \textbf{(243)}$$

the equation of motion is

$subs(\sin(\phi(t)) = \phi(t), \textbf{(242)})$

$$\ddot{\phi}\, l^2\, m + \phi\, g\, l\, m = 0 \quad \textbf{(244)}$$

For generic initial conditions,

$\phi(t_0) = \phi_0, \%eval(diff(\text{phi}(t), t), t = t_0) = \omega_0$

$$\phi(t_0) = \phi_0, \dot{\phi}\Big|_{t = t_0} = \omega_0 \quad \textbf{(245)}$$

we get

$dsolve([\textbf{(244)}, \textbf{(245)}])$

$$\phi = \frac{\left(\sqrt{g}\, \sin\left(\frac{\sqrt{g}\, t_0}{\sqrt{l}}\right) \phi_0 + \sqrt{l}\, \cos\left(\frac{\sqrt{g}\, t_0}{\sqrt{l}}\right) \omega_0\right) \sin\left(\frac{\sqrt{g}\, t}{\sqrt{l}}\right)}{\sqrt{g}} + \frac{\left(\phi_0 \sqrt{g}\, \cos\left(\frac{\sqrt{g}\, t_0}{\sqrt{l}}\right) - \omega_0 \sqrt{l}\, \sin\left(\frac{\sqrt{g}\, t_0}{\sqrt{l}}\right)\right) \cos\left(\frac{\sqrt{g}\, t}{\sqrt{l}}\right)}{\sqrt{g}} \quad \textbf{(246)}$$

Conservation laws

Work

By definition, the work performed by a force $\vec{F}$ to move a particle an infinitesimal amount $d\vec{r}$ is $\vec{F} \cdot d\vec{r}$. Thus, the work to move it from $\vec{A}$ to $\vec{B}$ along some path C is

$$\left(\int_{\vec{A}}^{\vec{B}} \vec{F} \cdot \mathrm{d}\vec{r}\right)_{path = C}$$

Problem

A particle is submitted to a force whose Cartesian components are given by $F_x = a\, x^3 + b\, x\, y^2 + c\, z$, $F_y = a\, y^3 + b\, x\, y^2$, $F_z = c\, z$. Calculate the work done by this force when moving the particle along a straight line from the origin to a point (x_0, y_0, z_0).

Solution

The work to be calculated is equal to the line integral of the force along the path (line) indicated. The steps to solve this problem are:

1. Determine the vectorial (parameter *t*) equation $\vec{r}(t)$ of the line over which we are going to integrate.

2. Express *x*, *y*, and *z* entering the components of $\vec{F}$ in terms of the components of $\vec{r}(t)$, after which the scalar product $\vec{F} \cdot d\vec{r}$ becomes $\vec{F}(t) \cdot \dot{\vec{r}}(t)\ dt$ and the work can be expressed as $\int_0^t \vec{F} \cdot \vec{v}\, dt$.

3. Compute the integral.

The same computation can be done in one go by asking the computer to compute the line integral (see at the end).

restart :
with(*Physics*:-*Vectors*) :

The work to be calculated is given by
$W := Int(F\_.v\_, t = 0..t)$

$$W := \int_0^t \vec{F} \cdot \vec{v}\, dt \quad (247)$$

The force acting on the particle is
$F\_ := (a\, x^3 + b\, x\, y^2 + c\, z)\_i + (a\, x^3 + b\, x\, y^2)\_j + c\, z\, \_k$

$$\vec{F} := (a\, x^3 + b\, x\, y^2 + c\, z)\, \hat{i} + (a\, x^3 + b\, x\, y^2)\, \hat{j} + c\, z\, \hat{k} \quad (248)$$

The vectorial equation of a line $\vec{r} := \vec{r}(t)$, where *t* is a parameter, is represented by the vector position of any of its points. This line passes through the origin and a generic point
$r\_[0] := x_0\, \_i + y_0\, \_j + z_0\, \_k$

$$\vec{r}_0 := x_0\, \hat{i} + y_0\, \hat{j} + z_0\, \hat{k} \quad (249)$$

In this case, $\vec{r}(t)$ can be written directly as the product of a parameter *t* and $\vec{r}_0$
$r\_ := t\ r\_[0]$

$$\vec{r} := t\,(x_0\, \hat{i} + y_0\, \hat{j} + z_0\, \hat{k}) \quad (250)$$

To construct the scalar product $\vec{F} \cdot d\vec{r}$, first express $\vec{F}$ in terms of the vectorial equation of this line; that is, *x*, *y*, and *z* entering its components shall be taken from the components of the equation above for $\vec{r}$
$x = Component(r\_, 1), y = Component(r\_, 2), z = Component(r\_, 3)$

$$x = t\, x_0, y = t\, y_0, z = t\, z_0 \quad (251)$$

$F\_ := eval(F\_, [\textbf{(251)}])$

$$\vec{F} := (a\, t^3\, x_0^{\,3} + b\, t^3\, x_0\, y_0^{\,2} + c\, t\, z_0)\, \hat{i} + (a\, t^3\, x_0^{\,3} + b\, t^3\, x_0\, y_0^{\,2})\, \hat{j} + c\, t\, z_0\, \hat{k} \quad (252)$$

Now compute $\vec{v} = \dot{\vec{r}}(t)$
$v\_ := diff(r\_, t)$

$$\vec{v} := x_0\, \hat{i} + y_0\, \hat{j} + z_0\, \hat{k} \quad (253)$$

So the line integral for the work W is now

W

$$\int_0^t \left(a t^3 x_0^4 + a t^3 x_0^3 y_0 + b t^3 x_0^2 y_0^2 + b t^3 x_0 y_0^3 + c t x_0 z_0 + c t z_0^2\right) dt \qquad \textbf{(254)}$$

At the origin, $t = 0$, and when the particle is at $\vec{r}_0$ we have $t = 1$, so the value of W to move the particle to $\vec{r}_0$ is

$eval(W, t = 1)$

$$\int_0^1 \left(a t^3 x_0^4 + a t^3 x_0^3 y_0 + b t^3 x_0^2 y_0^2 + b t^3 x_0 y_0^3 + c t x_0 z_0 + c t z_0^2\right) dt \qquad \textbf{(255)}$$

$value(\textbf{(255)})$

$$\frac{1}{4} a x_0^4 + \frac{1}{4} a x_0^3 y_0 + \frac{1}{4} b x_0^2 y_0^2 + \frac{1}{4} b x_0 y_0^3 + \frac{1}{2} c x_0 z_0 + \frac{1}{2} c z_0^2 \qquad \textbf{(256)}$$

The same computation can be done in one go by asking the system to compute the line integral directly. Two different ways of performing this computation are:

First, the parametric equations of the line that goes from 0 to $x_0 \hat{i} + y_0 \hat{j} + z_0 \hat{k}$ are

$C := \{x = t x_0, y = t y_0, z = t z_0\}$

$$C := \{x = t x_0, y = t y_0, z = t z_0\} \qquad \textbf{(257)}$$

This parametrization can also be computed using $ParametrizeCurve(line(0, [x_0, y_0, z_0]))$. The force is as shown in **(248)**

$F\_$

$$\left(a t^3 x_0^3 + b t^3 x_0 y_0^2 + c t z_0\right) \hat{i} + \left(a t^3 x_0^3 + b t^3 x_0 y_0^2\right) \hat{j} + c t z_0 \hat{k} \qquad \textbf{(258)}$$

The integration limits are

$A\_ := 0;$

$B\_ := x_0 \hat{i} + y_0 \hat{j} + z_0 \hat{k}$

$$\vec{A} := 0$$

$$\vec{B} := x_0 \hat{i} + y_0 \hat{j} + z_0 \hat{k} \qquad \textbf{(259)}$$

So this is the integral we want to compute

$Int('F\_, dr\_ = A\_ .. B\_', path = C)$

$$\left(\int_{\vec{A}}^{\vec{B}} \vec{F} \cdot d\vec{r}\right)_{path = \{x = t x_0, y = t y_0, z = t z_0\}} \qquad \textbf{(260)}$$

$value(\textbf{(260)})$

$$\frac{1}{4} a x_0^4 + \frac{1}{4} a x_0^3 y_0 + \frac{1}{4} b x_0^2 y_0^2 + \frac{1}{4} b x_0 y_0^3 + \frac{1}{2} c x_0 z_0 + \frac{1}{2} c z_0^2 \qquad \textbf{(261)}$$

This is the same result **(256)** achieved interactively while performing one step at a time

(261) − **(256)**

$$0 \qquad \textbf{(262)}$$

Since the path is a straight line connecting $\vec{A}$ and $\vec{B}$, the following line integral can also be entered

Int('*F_*, *dr_* = *A_* .. *B_*, *path* = *line*(*A_*, *B_*)')

$$\left(\int_{\vec{A}}^{\vec{B}} \vec{F} \cdot \mathrm{d}\vec{r}\right)_{path = line(\vec{A}, \vec{B})} \tag{263}$$

value((**263**))

$$\frac{1}{4} a x_0^4 + \frac{1}{4} a x_0^3 y_0 + \frac{1}{4} b x_0^2 y_0^2 + \frac{1}{4} b x_0 y_0^3 + \frac{1}{2} c x_0 z_0 + \frac{1}{2} c z_0^2 \tag{264}$$

Conservation of the total energy of a closed system or system in a constant external field

Problem

Consider a closed system, or a system in a constant external field, for which the total force acting on the i^{th} particle of the system can be derived from a potential, $\vec{F}_i = -\nabla_i U$. Show that the total *energy* of the system is conserved.

Solution

restart :
with(*Physics*:-*Vectors*) :
CompactDisplay((*x*, *y*, *z*, *v_*, *F_*)(*t*)) :

x(*t*) *will now be displayed as x*

y(*t*) *will now be displayed as y*

z(*t*) *will now be displayed as z*

$\vec{v}(t)$ *will now be displayed as* $\vec{v}$

$\vec{F}(t)$ *will now be displayed as* $\vec{F}$ **(265)**

For these systems, the kinetic and potential energies T and U respectively depend on the time t only through the velocity $\vec{v}_i(t)$ and $\vec{r}_i(t)$. The total energy $E = T + U$ expressed here as a function of the time (only through $\vec{v}_i(t)$ and $\vec{r}_i(t)$) for the purpose of showing that *the total energy of a closed system is constant*.

$E(t) = T(t) + U(t)$

$$E(t) = T(t) + U(t) \tag{266}$$

Since the kinetic energy T is additive, without loss of generality, to perform the computation we consider here the case of a system of one particle. Substituting T and U as functions of the velocity and the coordinates, we have

$$subs\left(T(t) = \frac{1}{2} m\, v\_(t) \,.\, v\_(t), U(t) = U(x(t), y(t), z(t)), \mathbf{(266)}\right)$$

$$E(t) = \frac{m \|\vec{v}\|^2}{2} + U(x, y, z) \quad \textbf{(267)}$$

Taking the total derivative with respect to time
$diff(\textbf{(267)}, t)$

$$\dot{E}(t) = m\,(\dot{\vec{v}} \cdot \vec{v}) + D_1(U)(x, y, z)\,\dot{x} + D_2(U)(x, y, z)\,\dot{y} + D_3(U)(x, y, z)\,\dot{z} \quad \textbf{(268)}$$

From Newton's 2nd law,
$diff(v\_(t), t) = \frac{F\_(t)}{m}$

$$\dot{\vec{v}} = \frac{\vec{F}}{m} \quad \textbf{(269)}$$

$subs(\textbf{(269)}, \textbf{(268)})$

$$\dot{E}(t) = m\left(\left(\frac{\vec{F}}{m}\right) \cdot \vec{v}\right) + D_1(U)(x, y, z)\,\dot{x} + D_2(U)(x, y, z)\,\dot{y} + D_3(U)(x, y, z)\,\dot{z} \quad \textbf{(270)}$$

In turn, in view of $\vec{F} = -\nabla U$, compute ∇U
$(Gradient = \%Gradient)(U(x(t), y(t), z(t)))$

$$D_1(U)(x, y, z)\,\hat{i} + D_2(U)(x, y, z)\,\hat{j} + D_3(U)(x, y, z)\,\hat{k} = \nabla U(x, y, z) \quad \textbf{(271)}$$

The position vector is
$r\_ := x(t) \cdot \_i + y(t) \cdot \_j + z(t) \cdot \_k$

$$\vec{r} := x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \quad \textbf{(272)}$$

from where $\nabla U \cdot \vec{v}$, expressed on the left-hand side in terms of $\dot{\vec{r}}(t)$ is equal to
$lhs(\textbf{(271)}).\,diff(r\_, t) = rhs(\textbf{(271)})\,.\,v\_(t)$

$$D_1(U)(x, y, z)\,\dot{x} + D_2(U)(x, y, z)\,\dot{y} + D_3(U)(x, y, z)\,\dot{z} = (\nabla U(x, y, z)) \cdot \vec{v} \quad \textbf{(273)}$$

Simplifying $\dot{E}(t)$ given in **(270)** using this expression **(273)** (alternatively, use algsubs)
$simplify(\textbf{(270)}, \{\textbf{(273)}\})$

$$\dot{E}(t) = \vec{F} \cdot \vec{v} + (\nabla U(x, y, z)) \cdot \vec{v} \quad \textbf{(274)}$$

Finally, from
$\%Gradient(U(x(t), y(t), z(t))) = -F\_(t)$

$$\nabla U(x, y, z) = -\vec{F} \quad \textbf{(275)}$$

$eval(\textbf{(274)}, \textbf{(275)})$

$$\dot{E}(t) = 0 \quad \textbf{(276)}$$

which is the expected result: the total energy of a closed system is conserved.

Problem
Consider a system of n particles in two reference systems K and K' that move relative to each other with constant velocity $\vec{V}$. Show that the relation between the energies of the system, E and E', is given by

$$E = E' + \frac{1}{2}\|\vec{V}\|^2\left(\sum_{a=1}^{n} m_a\right) + \vec{V} \cdot \vec{P'}$$

Solution

```
restart;
with(Physics:-Vectors) :
```

The energy E of the system, measured in K, is given by

$E = \frac{1}{2} Sum\left(m[a]\, v\_[a]^2,\ a = 1\,..\,n\right) + U$

$$E = \frac{\sum_{a=1}^{n} m_a \vec{v}_a^{\,2}}{2} + U \quad \textbf{(277)}$$

where U is the potential energy. From the composition of velocities, $\vec{v}_a$ in K is related to $\vec{v'}_a$ in K' by

$v\_[a] = V\_ + v'\_[a]$

$$\vec{v}_a = \vec{V} + \vec{v'}_a \quad \textbf{(278)}$$

$subs(\textbf{(278)}, \textbf{(277)})$

$$E = \frac{\sum_{a=1}^{n} m_a \left(\vec{V} + \vec{v'}_a\right)^2}{2} + U \quad \textbf{(279)}$$

Expanding the power and the Sum all at once

$expand(\textbf{(279)})$

$$E = \frac{\|\vec{V}\|^2 \left(\sum_{a=1}^{n} m_a\right)}{2} + \vec{V} \cdot \left(\sum_{a=1}^{n} m_a \vec{v'}_a\right) + \frac{\sum_{a=1}^{n} m_a \|\vec{v'}_a\|^2}{2} + U \quad \textbf{(280)}$$

Introducing the total momentum $\sum_{a=1}^{n} m_a \vec{v'}_a = \vec{P'}$ (copy from **(280)** and paste in the next input line)

$subs\left(\sum_{a=1}^{n} m_a \vec{v'}_a = P'\_, \textbf{(280)}\right)$

$$E = \frac{\|\vec{V}\|^2 \left(\sum_{a=1}^{n} m_a\right)}{2} + \vec{V} \cdot \vec{P'} + \frac{\sum_{a=1}^{n} m_a \|\vec{v'}_a\|^2}{2} + U \quad \textbf{(281)}$$

The remaining term of **(280)** that involves $\vec{v'}_a$ adds with U to represent the total energy E' in the the system K'

$simplify\left(\textbf{(281)}, \left\{\frac{\sum_{a=1}^{n} m_a Norm\left(v'\_{a}\right)^2}{2} + U = `E'`\right\}\right)$

$$E = E' + \frac{\|\vec{V}\|^2 \left(\sum_{a=1}^{n} m_a\right)}{2} + \vec{V} \cdot \vec{P'} \quad \textbf{(282)}$$

This is already the expected result. If the system is at rest in K', then $\overrightarrow{P'} = 0$, $E = E' + \frac{\|\vec{V}\|^2 \left(\sum_{a=1}^{n} m_a\right)}{2}$, and E' represents the system's *internal energy*.

Conservation of the total momentum of a closed system

The conservation of the total momentum of a closed system of one particle is clear: if the [illegible] does not [illegible] with anything external, the force acting on it is zero, and from Newton's 2nd law $\vec{F} = \dot{\vec{p}}(t)$ follows $\dot{\vec{p}}(t) = 0$.

For a closed system of many particles, while the total force acting on the system is equal to 0, there can be internal forces different from zero acting on each particle due to its interaction with the other particles. These internal forces, however, produce no acceleration of the system; in general, they cancel each other out due to Newton's 3rd law.

Problem
Consider a system of n particles measured in two frames of reference K and K' that move relative to each other with velocity $\vec{V}$. Show that the system's momenta $\vec{P}$ and $\vec{P}'$ are related by

$$\vec{P} = \vec{P}' + \vec{V} \cdot \sum_{a=1}^{n} m_a .$$

Solution

restart;
with(*Physics*:-*Vectors*) :

The momentum $\vec{P}$ of the system, measured in K, is given by
*P_* = *Sum*(*m*[*a*] *v_*[*a*], *a* = 1 .. *n*)

$$\vec{P} = \sum_{a=1}^{n} m_a \vec{v}_a \tag{283}$$

From the composition of velocities, $\vec{v}_a$ in K is related to $\vec{v'}_a$ in K' by
*v_*[*a*] = *V_* + *v'_*[*a*]

$$\vec{v}_a = \vec{V} + \vec{v'}_a \tag{284}$$

subs(**(284)**, **(283)**)

$$\vec{P} = \sum_{a=1}^{n} m_a \left(\vec{V} + \vec{v'}_a\right) \tag{285}$$

expand(**(285)**)

$$\vec{P} = \vec{V}\left(\sum_{a=1}^{n} m_a\right) + \left(\sum_{a=1}^{n} m_a \vec{v'}_a\right) \quad \textbf{(286)}$$

Introducing $\sum_{a=1}^{n} m_a \vec{v'}_a = \vec{P'}$

$subs\left(\sum_{a=1}^{n} m_a \vec{v'}_a = \vec{P'}, \textbf{(286)}\right)$

$$\vec{P} = \vec{V}\left(\sum_{a=1}^{n} m_a\right) + \vec{P'} \quad \textbf{(287)}$$

which is the desired relationship between $\vec{P}$ and $\vec{P'}$.

Problem

A particle of mass m moving with velocity $\vec{v}_1$ leaves a half-space in which its potential energy is a constant U_1 and enters another in which its potential energy is a different constant U_2. Determine the change in direction of motion of the particle; that is, $\frac{\sin(\theta_1)}{\sin(\theta_2)}$ where θ_1 and θ_2 are the angles between an axis perpendicular to the separating plane and the momentum $\vec{p}$ in the regions 1 and 2.

Solution

restart :
with(*Physics*:-*Vectors*) :

Consider a Cartesian frame with the axis z perpendicular to the plane that divides the two regions. Call $\vec{p}_1$ and $\vec{p}_2$ the momentum above and below the plane, call θ_1 the angle between $\vec{p}_1$ and the z vertical axis and θ_2 the angle between $\vec{p}_2$ and that same axis.

In this reference system, the potential energy U depends only on z, with constant values U_1 and U_2 above and below the separating plane such that $0 = \frac{\partial U}{\partial x} = \frac{\partial U}{\partial y}$. Therefore, the x and y components of the momentum are conserved. For simplicity we orient the y axis so that the component of the momentum in that direction is (and remains) zero. We have for $\vec{p}_1$ and $\vec{p}_2$

$p_{1\_} = Norm(p_{1\_}) \cdot \sin(\theta_1) \cdot \_i + Norm(p_{1\_}) \cdot \cos(\theta_1) \cdot \_k$

$$\vec{p}_1 = \|\vec{p}_1\| \sin(\theta_1)\,\hat{i} + \|\vec{p}_1\| \cos(\theta_1)\,\hat{k} \quad \textbf{(288)}$$

$p_{2\_} = Norm(p_{2\_}) \cdot \sin(\theta_2) \cdot \_i + Norm(p_{2\_}) \cdot \cos(\theta_2) \cdot \_k$

$$\vec{p}_2 = \|\vec{p}_2\| \sin(\theta_2)\,\hat{i} + \|\vec{p}_2\| \cos(\theta_2)\,\hat{k} \quad \textbf{(289)}$$

The projection of $\vec{p}_1$ and $\vec{p}_2$ over the plane is conserved, so

$(rhs(\textbf{(288)}) = rhs(\textbf{(289)})).\_i$

$$\|\vec{p}_1\| \sin(\theta_1) = \|\vec{p}_2\| \sin(\theta_2) \quad \textbf{(290)}$$

Rewrite **(290)** as the ratio of sines we want to determine

$$\frac{isolate(\textbf{(290)}, \sin(\theta_1))}{\sin(\theta_2)}$$

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{\|\vec{p}_2\|}{\|\vec{p}_1\|} \quad \textbf{(291)}$$

The total energy $E = T + U$ above and below the plane that separates the two regions is also conserved, so

$$\frac{Norm(p_{1\_})^2}{2\, m} + U_1 = \frac{Norm(p_{2\_})^2}{2\, m} + U_2$$

$$\frac{\|\vec{p}_1\|^2}{2\, m} + U_1 = \frac{\|\vec{p}_2\|^2}{2\, m} + U_2 \quad \textbf{(292)}$$

We can use **(292)** to eliminate the value of the momentum in one of the two regions by first squaring **(291)**

$\textbf{(291)}^2$

$$\frac{\sin(\theta_1)^2}{\sin(\theta_2)^2} = \frac{\|\vec{p}_2\|^2}{\|\vec{p}_1\|^2} \quad \textbf{(293)}$$

Now simplify **(293)** by using **(292)** to eliminate the norm of - say - $\vec{p}_2$

$simplify(\textbf{(293)}, \{\textbf{(292)}\}, \{Norm(p_{2\_})\})$

$$\frac{\sin(\theta_1)^2}{\sin(\theta_2)^2} = \frac{\|\vec{p}_1\|^2 + 2\, m\, (U_1 - U_2)}{\|\vec{p}_1\|^2} \quad \textbf{(294)}$$

Collecting m and $\|\vec{p}_1\|$

$$collect\left(\textbf{(294)}, \left[m, \frac{1}{Norm(p_{1\_})^2}\right]\right)$$

$$\frac{\sin(\theta_1)^2}{\sin(\theta_2)^2} = 1 + \frac{(2\, U_1 - 2\, U_2)\, m}{\|\vec{p}_1\|^2} \quad \textbf{(295)}$$

By computing the square root of each side of **(295)** while taking into account that $\sin(\theta_1) > 0, \sin(\theta_2) > 0$, we get the change of direction of the particle when it passes through the plane that separates the two regions as a function of the jump $U_1 - U_2$ of the potential energy and the absolute value of momentum $\|\vec{p}_1\|$

$map(\text{sqrt}, \textbf{(295)})$ assuming $\sin(\theta_1) > 0, \sin(\theta_2) > 0$

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \sqrt{1 + \frac{(2\, U_1 - 2\, U_2)\, m}{\|\vec{p}_1\|^2}} \quad \textbf{(296)}$$

Conservation of angular momentum

Like the conservation of *linear* momentum, the conservation of the total *angular* momentum of a closed system of one particle is natural: if the particle does not interact with anything external, the force acting on it is zero and therefore its torque $\vec{N} = \vec{r} \times \vec{F} = 0$. From $\dot{\vec{L}} = \vec{N}$ it follows that $\dot{\vec{L}} = 0$, that is, the angular momentum $\vec{L}$ is conserved.

Problem

a) Express the Cartesian components of the angular momentum $\vec{L}$, as well as its norm, in cylindrical and spherical coordinates.
b) Rewrite $\vec{L}$ in cylindrical coordinates and using the cylindrical orthonormal basis of unit vectors, then do the same using spherical coordinates and the spherical basis.

Solution

restart;
with(*Physics*:-*Vectors*) :

a) It is simpler to work with vectorial expressions so as to accomplish the transformation of the three Cartesian components at once.

First set a compact display for all the coordinates as functions of t
$CompactDisplay((x, y, z, \rho, \phi, r, \theta, \_\rho, \_\phi, \_\theta)(t))$

$$x(t) \textit{ will now be displayed as } x$$
$$y(t) \textit{ will now be displayed as } y$$
$$z(t) \textit{ will now be displayed as } z$$
$$\rho(t) \textit{ will now be displayed as } \rho$$
$$\phi(t) \textit{ will now be displayed as } \phi$$
$$r(t) \textit{ will now be displayed as } r$$
$$\theta(t) \textit{ will now be displayed as } \theta$$
$$\hat{\rho}(t) \textit{ will now be displayed as } \hat{\rho}$$
$$\hat{\phi}(t) \textit{ will now be displayed as } \hat{\phi}$$
$$\hat{\theta}(t) \textit{ will now be displayed as } \hat{\theta} \quad \textbf{(297)}$$

The angular momentum,
$L\_ = r\_ \times p\_$

$$\vec{L} = \vec{r} \times \vec{p} \quad \textbf{(298)}$$

The position in Cartesian coordinates,
$r\_ = x(t)\,\_i + y(t)\,\_j + z(t)\,\_k$

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \tag{299}$$

$Assume(m > 0)$

$$\{m::(0, \infty)\} \tag{300}$$

The momentum:

$p\_ = m \cdot diff(rhs(\mathbf{(299)}), t)$

$$\vec{p} = m\,(\dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k}) \tag{301}$$

from which

$subs(\mathbf{(299)}, \mathbf{(301)}, \mathbf{(298)})$

$$\vec{L} = (x\,\hat{i} + y\,\hat{j} + z\,\hat{k}) \times (m\,(\dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k})) \tag{302}$$

Performing the cross product

(302)

$$\vec{L} = m\,(y\,\dot{z} - z\,\dot{y})\,\hat{i} - m\,(\dot{z}\,x - z\,\dot{x})\,\hat{j} - m\,(y\,\dot{x} - \dot{y}\,x)\,\hat{k} \tag{303}$$

To combine the two steps into one, in **(302)** use *eval* instead of *subs*. The simplest approach to expressing the Cartesian components of **(303)** in cylindrical coordinates is to use the ChangeCoordinates command

$ChangeCoordinates(\mathbf{(302)}, cylindrical)$

$$\vec{L} = m\,(-z\,\rho\cos(\phi)\,\dot{\phi} + \sin(\phi)\,(\dot{z}\,\rho - z\,\dot{\rho}))\,\hat{i} + m\,(-\rho\sin(\phi)\,\dot{\phi}\,z - \cos(\phi)\,(\dot{z}\,\rho - z\,\dot{\rho}))\,\hat{j} + m\,\dot{\phi}\,\rho^2\,\hat{k} \tag{304}$$

So for instance $L_x = \vec{L} \cdot \hat{i}$

$\mathbf{(304)}\,.\,\_i$

$$\vec{L} \cdot \hat{i} = m\,(-z\,\rho\cos(\phi)\,\dot{\phi} + \sin(\phi)\,(\dot{z}\,\rho - z\,\dot{\rho})) \tag{305}$$

For the square of the norm of $\vec{L}$

$Norm(\mathbf{(304)})^2$

$$\|\vec{L}\|^2 = m^2\,(-z\,\rho\cos(\phi)\,\dot{\phi} + \sin(\phi)\,(\dot{z}\,\rho - z\,\dot{\rho}))^2 + m^2\,(-\rho\sin(\phi)\,\dot{\phi}\,z - \cos(\phi)\,(\dot{z}\,\rho - z\,\dot{\rho}))^2 + m^2\,\dot{\phi}^2\,\rho^4 \tag{306}$$

$simplify(\mathbf{(306)})$

$$\|\vec{L}\|^2 = m^2\,((z^2\,\rho^2 + \rho^4)\,\dot{\phi}^2 + (z\,\dot{\rho} - \dot{z}\,\rho)^2) \tag{307}$$

---

b) To rewrite the expression **(302)** of $\vec{L}$ in the cylindrical orthonormal basis and using cylindrical coordinates, use ChangeBasis with the optional argument *alsothecoordinates*

$lhs(\mathbf{(302)}) = ChangeBasis(rhs(\mathbf{(302)}), cylindrical, also)$

* *Partial match of 'also' against keyword 'alsothecoordinates'*

$$\vec{L} = -m\,\dot{\phi}\,\rho\,z\,\hat{\rho} + m\,(z\,\dot{\rho} - \dot{z}\,\rho)\,\hat{\phi} + m\,\dot{\phi}\,\rho^2\,\hat{k} \tag{308}$$

Naturally, the norm of $\vec{L}$ is the same regardless of the orthonormal basis in which the vector is expressed

$Norm(\mathbf{(308)})^2$

$$\|\vec{L}\|^2 = m^2\,\dot{\phi}^2\,\rho^2\,z^2 + m^2\,(z\,\dot{\rho} - \dot{z}\,\rho)^2 + m^2\,\dot{\phi}^2\,\rho^4 \tag{309}$$

Compare with **(307)**

$simplify((\mathbf{307}) - (\mathbf{309}))$

$$0 = 0 \tag{310}$$

The steps in spherical coordinates are the same.

a)

$ChangeCoordinates((\mathbf{302}), spherical)$

$$\vec{L} = -r^2 m (\sin(\theta)\cos(\phi)\cos(\theta)\dot{\phi} + \dot{\theta}\sin(\phi))\hat{i} - r^2 m (\sin(\theta)\cos(\theta)\sin(\phi)\dot{\phi} - \dot{\theta}\cos(\phi))\hat{j} - r^2 m (\cos(\theta)^2 - 1)\dot{\phi}\hat{k} \tag{311}$$

$(\mathbf{311}) . \_i$

$$\vec{L} \cdot \hat{i} = -r^2 m (\sin(\theta)\cos(\phi)\cos(\theta)\dot{\phi} + \dot{\theta}\sin(\phi)) \tag{312}$$

$Norm((\mathbf{311}))^2$

$$\|\vec{L}\|^2 = r^4 m^2 (\sin(\theta)\cos(\phi)\cos(\theta)\dot{\phi} + \dot{\theta}\sin(\phi))^2 + r^4 m^2 (\sin(\theta)\cos(\theta)\sin(\phi)\dot{\phi} - \dot{\theta}\cos(\phi))^2 + r^4 m^2 \dot{\phi}^2 \sin(\theta)^4 \tag{313}$$

$simplify((\mathbf{313}))$

$$\|\vec{L}\|^2 = -r^4 m^2 ((\cos(\theta)^2 - 1)\dot{\phi}^2 - \dot{\theta}^2) \tag{314}$$

---

b)

$lhs((\mathbf{302})) = ChangeBasis(rhs((\mathbf{302})), spherical, also)$

** Partial match of 'also' against keyword 'alsothecoordinates'*

$$\vec{L} = -m r^2 \dot{\phi}\sin(\theta)\hat{\theta} + m r^2 \hat{\phi}\dot{\theta} \tag{315}$$

$Norm((\mathbf{315}))^2$

$$\|\vec{L}\|^2 = m^2 r^4 \dot{\phi}^2 \sin(\theta)^2 + m^2 r^4 \dot{\theta}^2 \tag{316}$$

Problem

Consider a system of n particles measured in two frames of reference K and K' whose origins have distance $\vec{A}$ from each other. Show that the momenta $\vec{L}$ and $\vec{L}'$ of the system are related by

$$\vec{L} = \vec{L}' + \vec{A} \times \vec{P}$$

where $\vec{P} = \sum_{a=1}^{n} \vec{p}_a$ is the total momentum of the system as seen from K.

Solution

restart;
$with(Physics\text{:-}Vectors):$

The momentum $\vec{P}$ of the system, measured in K, is given by

$L\_ = Sum(r\_[a] \times p\_[a],\ a = 1 .. n)$

$$\vec{L} = \sum_{a=1}^{n} \vec{r}_a \times \vec{p}_a \quad \textbf{(317)}$$

The relationship between $\vec{r}_a$ and $\vec{r'}_a$ measured in K and K' is given by

$r\_[a] = A\_ + r'\_[a]$

$$\vec{r}_a = \vec{A} + \vec{r'}_a \quad \textbf{(318)}$$

$subs(\textbf{(318)}, \textbf{(317)})$

$$\vec{L} = \sum_{a=1}^{n} \left(\vec{A} + \vec{r'}_a\right) \times \vec{p}_a \quad \textbf{(319)}$$

Expanding

$expand(\textbf{(319)})$

$$\vec{L} = \vec{A} \times \left(\sum_{a=1}^{n} \vec{p}_a\right) + \left(\sum_{a=1}^{n} \vec{r'}_a \times \vec{p}_a\right) \quad \textbf{(320)}$$

Since K and K' are at rest with respect to each other, $\vec{v} = \vec{v'}$ and so $\vec{p} = \vec{p'}$. Hence, $\left(\sum_{a=1}^{n} \vec{r'}_a \times \vec{p}_a\right) = \vec{L'}$, the angular momentum of the system in K'. The expression $\sum_{a=1}^{n} \vec{p}_a = \vec{P}$ is the linear momentum of the system in K (copy the *sum* subexpressions from **(320)**, paste, then edit). Taking both of these into account,

$subs\left(\left(\sum_{a=1}^{n} \vec{r'}_a \times \vec{p}_a\right) = L'\_, \sum_{a=1}^{n} \vec{p}_a = \vec{P}, \textbf{(320)}\right)$

$$\vec{L} = \vec{A} \times \vec{P} + \vec{L'} \quad \textbf{(321)}$$

which is the desired relationship between $\vec{L}$ and $\vec{L'}$.

Problem

a) Consider a closed system of n particles, and two frames of reference K and K' that move relative to each other with a constant velocity $\vec{V}$. Show that the momenta $\vec{L}$ and $\vec{L}'$ respectively measured in K and K' are related by

$$\vec{L} = \left(\sum_{a=1}^{n} m_a\right)\left(\vec{R} \times \vec{V}\right) + \vec{A} \times \vec{P'} + \vec{L'}$$

where $\vec{A}$ is the distance from the origin of K to the origin of K', $\vec{R} = \sum_{=1}^{n} m_a \vec{r}_a / \sum_{a=1}^{n} m_a$ is the position of the center of mass as seen from K, and $\vec{P'}$ and $\vec{L'}$ are the total linear and angular momenta measured in K'.

b) Show that when the origin of K' is the center of mass $\vec{R}$, this formula reduces to

$$\vec{L} = \vec{L'} + \vec{R} \times \vec{P},$$

where

$\vec{P} = \sum_{a=1}^{n} m_a \vec{v}_a$ is the total linear momentum in K.

Solution

restart;
with(*Physics*:-*Vectors*) :

a) The momentum $\vec{L}$ of the system measured in K is given by
$L\_ = Sum(r\_[a] \times p\_[a],\ a = 1 .. n)$

$$\vec{L} = \sum_{a=1}^{n} \vec{r}_a \times \vec{p}_a \quad \textbf{(322)}$$

The right-hand side of the above can be expressed in terms of $\vec{P'}$ and $\vec{L'}$ using $\vec{p}_a = m_a \vec{v}_a$, where $\vec{v}_a = \vec{v'}_a + \vec{V}$:
$p\_[a] = p'\_[a] + m[a] \cdot V\_$

$$\vec{p}_a = m_a \vec{V} + \vec{p'}_a \quad \textbf{(323)}$$

subs(**(323)**, **(322)**)

$$\vec{L} = \sum_{a=1}^{n} \vec{r}_a \times \left(m_a \vec{V} + \vec{p'}_a\right) \quad \textbf{(324)}$$

ee := *expand*(**(324)**)

$$ee := \vec{L} = -\vec{V} \times \left(\sum_{a=1}^{n} m_a \vec{r}_a\right) + \left(\sum_{a=1}^{n} \vec{r}_a \times \vec{p'}_a\right) \quad \textbf{(325)}$$

The term with $\sum_{a=1}^{n} m_a \vec{r}_a$ can be expressed in terms of the position vector of the center of mass $\vec{R}$ (copy the subexpression from **(325)** , paste, then edit)

$$subs\left(\left(\sum_{a=1}^{n} m_a \vec{r}_a\right) = \left(\sum_{a=1}^{n} m_a\right) R\_, \textbf{(325)}\right)$$

$$\vec{L} = -\vec{V} \times \left(\left(\sum_{a=1}^{n} m_a\right) \vec{R}\right) + \left(\sum_{a=1}^{n} \vec{r}_a \times \vec{p'}_a\right) \quad \textbf{(326)}$$

To express $\sum_{a=1}^{n} \vec{r}_a \times \vec{p'}_a$ in terms of $\vec{L'}$ and $\vec{A}$, introduce the relation between $\vec{r}_a$ and $\vec{r'}_a$
$r\_[a] = A\_ + r'\_[a]$

$$\vec{r}_a = \vec{A} + \vec{r'}_a \quad \textbf{(327)}$$

subs(**(327)**, **(326)**)

$$\vec{L} = -\left(\sum_{a=1}^{n} m_a\right) \left(\vec{V} \times \vec{R}\right) + \left(\sum_{a=1}^{n} \left(\vec{A} + \vec{r'}_a\right) \times \vec{p'}_a\right) \quad \textbf{(328)}$$

expand((**328**))

$$\vec{L} = -\left(\sum_{a=1}^{n} m_a\right)(\vec{V} \times \vec{R}) + \vec{A} \times \left(\sum_{a=1}^{n} \vec{p'}_a\right) + \left(\sum_{a=1}^{n} \vec{r'}_a \times \vec{p'}_a\right) \quad \textbf{(329)}$$

On the right-hand side, two of the sums represent $\vec{P'}$ and $\vec{L'}$ (copy the *sum* subexpressions, paste into the next line, then edit)

$$subs\left(\left(\sum_{a=1}^{n} \vec{r'}_a \times \vec{p'}_a\right) = L'\_, \sum_{a=1}^{n} \vec{p'}_a = P'\_, \textbf{(329)}\right)$$

$$\vec{L} = -\left(\sum_{a=1}^{n} m_a\right)(\vec{V} \times \vec{R}) + \vec{A} \times \vec{P'} + \vec{L'} \quad \textbf{(330)}$$

This is already the desired result.

---

b) If the origin of K' is the center of mass $\vec{R}$, then $\vec{A} = \vec{R}$

subs($A\_ = R\_$, **(330)**)

$$\vec{L} = -\left(\sum_{a=1}^{n} m_a\right)(\vec{V} \times \vec{R}) + \vec{R} \times \vec{P'} + \vec{L'} \quad \textbf{(331)}$$

$\vec{P}$ and $\vec{P'}$ were related in **(287)** $\equiv$ $\vec{P} = \vec{V}\left(\sum_{a=1}^{n} m_a\right) + \vec{P'}$. This relation can be used to express $\vec{L}$ in terms of $\vec{P}$ instead of $\vec{P'}$

simplify((**331**), {(**287**)}, {$P'\_$})

$$\vec{L} = -\left(\sum_{a=1}^{n} m_a\right)(\vec{V} \times \vec{R}) + \vec{R} \times \left(-\vec{V}\left(\sum_{a=1}^{n} m_a\right) + \vec{P}\right) + \vec{L'} \quad \textbf{(332)}$$

expand((**332**))

$$\vec{L} = \vec{R} \times \vec{P} + \vec{L'} \quad \textbf{(333)}$$

which is the result we were looking for.

Cyclic coordinates

Any generalized coordinate q_i which does not appear explicitly in the Lagrangian is called *cyclic*. To any *cyclic* coordinate corresponds a conserved quantity. From

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

when q_i is cyclic, the right-hand side is 0 and so the quantity $\frac{\partial L}{\partial \dot{q}_i}$ is conserved.

Problem

The Lagrangian describing the movement of a particle in a central field has ϕ as a cyclic coordinate. Using cylindrical coordinates, show that the corresponding conserved quantity is the z component of the angular momentum $\vec{L}$

Solution

restart;
with(*Physics*:-*Vectors*) :

For convenience place the origin of the reference system at the center of the central field. Then the Lagrangian for a particle of mass m is

$L(t) = \frac{m}{2} \cdot diff(r\_(t), t)^2 - U(Norm(r\_))$

$$L(t) = \frac{m\, \dot{\vec{r}}(t)^2}{2} - U(\|\vec{r}\|) \quad \textbf{(334)}$$

Two rewrite the kinetic energy in terms of cylindrical coordinates, rewrite first in those coordinates the position vector $\vec{r}$

$r\_(t) = x(t) \cdot \_i + y(t) \cdot \_j + z(t) \cdot \_k$

$$\vec{r}(t) = x(t)\, \hat{i} + y(t)\, \hat{j} + z(t)\, \hat{k} \quad \textbf{(335)}$$

$lhs(\textbf{(335)}) = ChangeBasis(rhs(\textbf{(335)}), cylindrical, also)$

* *Partial match of 'also' against keyword 'alsothecoordinates'*

$$\vec{r}(t) = z(t)\, \hat{k} + \rho(t)\, \hat{\rho}(t) \quad \textbf{(336)}$$

$CompactDisplay(\textbf{(336)}, (L, p\_, \text{rho}, \_rho, \text{phi}, \_phi)(t))$

$L(t)$ *will now be displayed as* L

$\hat{\rho}(t)$ *will now be displayed as* $\hat{\rho}$

$\vec{r}(t)$ *will now be displayed as* $\vec{r}$

$\vec{p}(t)$ *will now be displayed as* $\vec{p}$

$\rho(t)$ *will now be displayed as* ρ

$z(t)$ *will now be displayed as* z

$\phi(t)$ *will now be displayed as* ϕ

$\hat{\phi}(t)$ *will now be displayed as* $\hat{\phi}$ **(337)**

One can now *subs*(*titute*) followed by evaluating the result (two steps), or do both steps in one go using *eval*(*uate*); in which case, after substitution, the derivative of $\hat{\rho}$ will also be performed

$eval(\textbf{(334)}, \textbf{(336)})$

$$L = \frac{m\, (\dot{z}\, \hat{k} + \dot{\rho}\, \hat{\rho} + \rho\, \dot{\phi}\, \hat{\phi})^2}{2} - U(\|\vec{r}\|) \quad \textbf{(338)}$$

Computing now $\frac{\partial L}{\partial \dot{q}_i}$, the conserved quantity is

$\%diff(lhs(\textbf{(338)}), diff(\text{phi}(t), t)) = diff(rhs(\textbf{(338)}), diff(\text{phi}(t), t))$

$$\frac{dL}{d\dot{\phi}} = m\,\rho^2\,\dot{\phi} \quad (339)$$

In turn, from the definition of angular momentum

$L\_(t) = r\_(t) \times p\_(t)$

$$\vec{L}(t) = \vec{r} \times \vec{p} \quad (340)$$

Introduce the velocity as the derivative of the position vector $\vec{r}$, then its expression in cylindrical coordinates **(336)** $\equiv \vec{r} = z\,\hat{k} + \hat{\rho}\,\rho$

$p\_(t) = m \cdot diff(r\_(t), t)$

$$\vec{p} = m\,\dot{\vec{r}} \quad (341)$$

$eval($**(340)**, **(341)**$)$

$$\vec{L}(t) = m\,(\vec{r} \times \dot{\vec{r}}) \quad (342)$$

$eval($**(342)**, **(336)**$)$

$$\vec{L}(t) = m\left(-z\,\rho\,\dot{\phi}\,\hat{\rho} + (z\,\dot{\rho} - \rho\,\dot{z})\,\hat{\phi} + \rho^2\,\dot{\phi}\,\hat{k}\right) \quad (343)$$

To get the Component in the direction of $\hat{k}$, you can input $Component($**(343)**$, 3)$ or directly project onto the z axis

(343) . $\_k$

$$\vec{L}(t) \cdot \hat{k} = m\,\rho^2\,\dot{\phi} \quad (344)$$

This result is the same as **(339)** computed differentiating the Lagrangian with respect to $\dot{\phi}$.

Integration of the equations of motion

Motion in one dimension

Problem

For a closed system, or any system where the total energy $E = T + U$ is conserved, show the following:

a) The trajectory in implicit form can always be computed directly from E.

b) The turning points, if any, can be computed directly from U.

Solution

```
restart;
with(Physics) :
```

The Lagrangian of a system with only one degree of freedom, represented by a generalized coordinate $q(t)$, is given by

$CompactDisplay(q(t))$

$$q(t) \textit{ will now be displayed as } q \quad (345)$$

$L = \frac{1}{2} a(q(t)) \cdot diff(q(t), t)^2 - U(q(t))$

$$L = \frac{a(q)\,\dot{q}^2}{2} - U(q) \tag{346}$$

a) The total energy $E = T + U$ is
$E = subs\left(U(q(t)) = -U(q(t)), rhs(\mathbf{(346)})\right)$

$$E = \frac{a(q)\,\dot{q}^2}{2} + U(q) \tag{347}$$

If the total energy E is known, this is a first order differential equation for $q(t)$, whose solution can be written in general form as
$dsolve(\mathbf{(347)})$

$$q = RootOf(2E - 2U(\_Z)),\; t - \left(\int^{q} \frac{a(\_a)\sqrt{2}}{2\sqrt{a(\_a)\,(E - U(\_a))}}\, \mathrm{d}\_a\right) - c_1 = 0,\; t - \left(\int^{q} -\frac{a(\_a)\sqrt{2}}{2\sqrt{a(\_a)\,(E - U(\_a))}}\, \mathrm{d}\_a\right) - c_1 = 0 \tag{348}$$

The first solution is a singular one
$\mathbf{(348)}[1]$

$$q = RootOf(2E - 2U(\_Z)) \tag{349}$$

One can verify this solves the constancy of the energy E viewed as an equation of movement
$odetest(\mathbf{(349)}, \mathbf{(347)})$

$$0 \tag{350}$$

This "singular" solution is however trivial: it corresponds to *no movement*. To see that, remove the RootOf
$DEtools{:}\text{-}remove\_RootOf(\mathbf{(349)})$

$$2E - 2U(q) = 0 \tag{351}$$

The above is true only when $T = 0$. The interesting solutions are the other two in **(348)**. For positive initial conditions $(c_1 > 0)$, from the other two solutions, the one for which $t > 0$ is the second one
$(t = t) - \mathbf{(348)}[2]$

$$\int^{q} \frac{a(\_a)\sqrt{2}}{2\sqrt{a(\_a)\,(E - U(\_a))}}\, \mathrm{d}\_a + c_1 = t \tag{352}$$

This is an implicit solution expressing $t \equiv t(q)$, that includes an arbitrary constant c_1 that depends on the initial value of q. When the integral can be computed, one could try solving the resulting expression to invert $t(q)$ and get an explicit solution $q(t)$.

---

b) Since the kinetic energy T is positive, $E = T + U > U$, from which the movement can only take place in regions where $U \le E$, with turning points where $U = E$.

Problem
Determine the period of oscillations of a pendulum of mass m and length l in a gravitational field as a

function of the amplitude of the oscillations

Solution

restart;
with(*Physics*) :

Working with spherical coordinates automatically sets their range
Coordinates(*spherical*)

$$\textit{Systems of spacetime coordinates are: } \{X = (r, \theta, \phi, t)\}$$

$$\{X\} \tag{353}$$

about(ϕ)

```
Originally phi, renamed phi:
  is assumed to be: RealRange(0,2*Pi)
```

The energy E of a pendulum in spherical coordinates is given by
CompactDisplay(phi(*t*))

$$\phi(t) \textit{ will now be displayed as } \phi \tag{354}$$

$E = \frac{1}{2} m \cdot l^2 \cdot diff(\text{phi}(t), t)^2 - m\, g\, l \cos(\text{phi}(t))$

$$E = \frac{m\, l^2\, \dot{\phi}^2}{2} - m\, g\, l \cos(\phi) \tag{355}$$

Set ranges for m and l to enable the computation of integrals further below
Assume($g > 0, l > 0, m > 0$)

$$\{g::(0, \infty)\}, \{l::(0, \infty)\}, \{m::(0, \infty)\} \tag{356}$$

Counting ϕ starting from $\phi\Big|_{t=0} = 0$,
dsolve([**(355)**, ϕ(0) = 0])

$$\phi = RootOf\left(t - \left(\int_0^{\_Z} -\frac{m\sqrt{2}\, l}{2\sqrt{m\,(m\, g\, l \cos(\_a) + E)}}\, \mathrm{d}\_a\right)\right), \phi = RootOf\left(t - \left(\int_0^{\_Z} \frac{m\sqrt{2}\, l}{2\sqrt{m\,(m\, g\, l \cos(\_a) + E)}}\, \mathrm{d}\_a\right)\right) \tag{357}$$

Following the reasoning used in the previous problem, take the solution for which $t > 0$, that is the second one
DEtools:-*remove_RootOf*(**(357)**[2])

$$t - \left(\int_0^{\phi} \frac{m\sqrt{2}\, l}{2\sqrt{m\,(m\, g\, l \cos(\_a) + E)}}\, \mathrm{d}\_a\right) = 0 \tag{358}$$

Calling

ϕ_{max} the maximum value of $\phi(t)$, the period of oscillations can be computed as the time t taken to go from $\phi = 0$ to $\phi = \phi_{max}$ multiplied by 4

$Assume(\phi_{max} > 0)$

$$\{\phi_{max} :: (0, \infty)\} \tag{359}$$

$isolate\left(subs\left(\left[t = \frac{T}{4}, \phi(t) = \phi_0\right], \textbf{(358)}\right), T\right)$

$$T = 4 \left(\int_0^{\phi_0} \frac{m\sqrt{2}\, l}{2\sqrt{m\,(m\, g\, l \cos(\_a) + E)}}\, \mathrm{d}\_a\right) \tag{360}$$

When $\dot{\phi} = 0$, the energy E, a constant, is equal to $-U(\phi_{max}) = -m\, g\, l \cos(\phi_{max})$

$simplify\left(subs\left(E = -m\, g\, l \cos(\phi_{max}), \textbf{(360)}\right)\right)$

$$T = \frac{2\sqrt{2}\sqrt{l}\left(\int_0^{\phi_0} \frac{1}{\sqrt{-\cos(\phi_{max}) + \cos(\_a)}}\, \mathrm{d}\_a\right)}{\sqrt{g}} \tag{361}$$

This integral can be computed in terms of the InverseJacobiAM

$value(\textbf{(361)})$

$$T = \frac{4\sqrt{2}\sqrt{l}\; am^{-1}\left(\frac{\phi_0}{2} \,\middle|\, \frac{\sqrt{2}}{\sqrt{-\cos(\phi_{max}) + 1}}\right)}{\sqrt{g}\sqrt{-\cos(\phi_{max}) + 1}} \tag{362}$$

where $am^{-1}(\phi|k) = \text{InverseJacobiAM}(\text{phi}, k)$. General information on special functions can be seen using the FunctionAdvisor

$FunctionAdvisor(definition, \text{InverseJacobiAM})$

$$\left[am^{-1}(\phi|k) = \int_0^{\phi} \frac{1}{\sqrt{1 - k^2 \sin(\_\theta 1)^2}}\, \mathrm{d}\_\theta 1, \textit{with no restrictions on } (\phi, k)\right] \tag{363}$$

$FunctionAdvisor(\text{InverseJacobiAM})$

InverseJacobiAM

describe

InverseJacobiAM = *trigonometric form of the incomplete elliptic integral of the first kind*

definition

$$am^{-1}(\phi|k) = \int_0^{\phi} \frac{1}{\sqrt{1 - k^2 \sin(\_\theta 1)^2}} \, d\_\theta 1$$ *with no restrictions on* (ϕ, k)

classify function

Elliptic_related

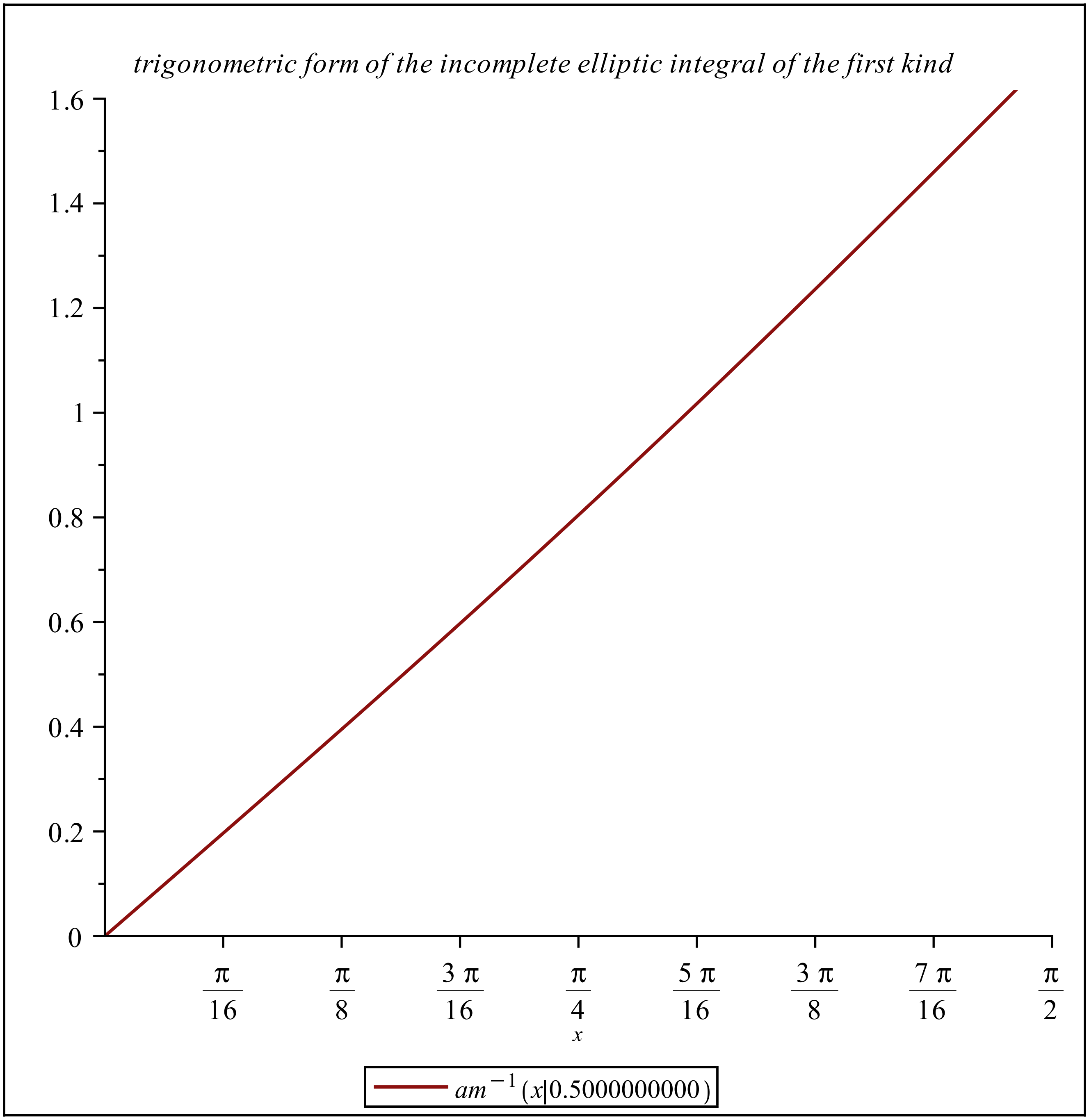
symmetries

$$am^{-1}(-\phi|k) = -am^{-1}(\phi|k)$$

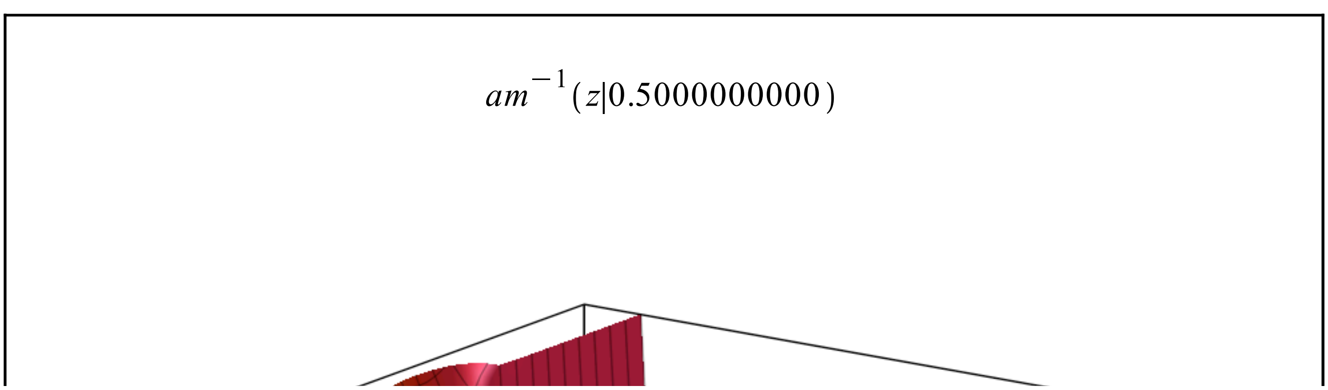
periodicity

$$am^{-1}(\phi|k)$$ No periodicity

plot



trigonometric form of the incomplete elliptic integral of the first kind
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
π/16
π/8
3 π/16
π/4
5 π/16
3 π/8
7 π/16
π/2
x
am^−1(x|0.5000000000)



am^−1(z|0.5000000000)

singularities

$am^{-1}(\phi \mid k)$	No isolated singularities

branch points

$am^{-1}(\phi \mid k)$	$\left(\phi = \arcsin\left(\frac{1}{k}\right) + n\,\pi \wedge n::\mathbb{Z}\right) \vee \left(\phi = -\arcsin\left(\frac{1}{k}\right) + n\,\pi \wedge n::\mathbb{Z}\right) \vee \phi = \infty + \infty\,\mathrm{I} \vee k = \csc(\phi) \vee k = -\csc(\phi) \vee$ $= \infty + \infty\,\mathrm{I}$

special values

$am^{-1}(-\phi \vert k) = -am^{-1}(\phi \vert k)$	
$am^{-1}(\phi \vert -k) = am^{-1}(\phi \vert k)$	
$am^{-1}(0 \vert k) = 0$	
$am^{-1}(\phi \vert 0) = \phi$	
$am^{-1}\left(\frac{n\pi}{2} \middle\vert k\right) = n\,\boldsymbol{K}(k)$	$n::\mathbb{Z}$
$am^{-1}(\phi \vert k) = \ln(\sec(\phi) + \tan(\phi))$	$\lvert\Re(\phi)\rvert < \frac{\pi}{2} \wedge k \in \{-1, 1\}$
$am^{-1}(\phi \vert k) = 0$	$k \in \{\infty, -\infty\}$
$am^{-1}(\infty\,\mathrm{I} \vert k) = \boldsymbol{K}(k) - \frac{\boldsymbol{K}\left(\frac{1}{k}\right)}{\sqrt{k}}$	$k^2 \in (0, 1)$

identities

$am^{-1}(\pi\,n + \phi \vert k) = 2\,n\,\boldsymbol{K}(k) + am^{-1}(\phi \vert k)$	$n::\mathbb{Z}$

sum form

$$am^{-1}(\phi|k) = \sum_{\_m2=0}^{\infty} \sum_{\_m1=0}^{\infty} \left((k^2)^{\_m1} \sin(\phi)^{2\,\_m2 + 2\,\_m1 + 1} \left(\frac{1}{2}\right)_{\_m1} \right) \Big/ \left((2\,\_m2 + 2\,\_m1 + 1)\,\_m2!\,\_m1! \right)$$

with no restrictions on (ϕ, k)

series

$$series(am^{-1}(\phi|k), \phi, 4) = \phi + \frac{1}{6} k^2 \phi^3 + O(\phi^4)$$

integral form

$$am^{-1}(\phi|k) = \int_0^{\phi} \frac{1}{\sqrt{1 - k^2 \sin(\_\theta 1)^2}} \, d\_\theta 1$$

with no restrictions on (ϕ, k)

differentiation rule

$$\frac{\partial}{\partial \phi} am^{-1}(\phi|k) = \frac{1}{\sqrt{1 - k^2 \sin(\phi)^2}}$$

$$\frac{\partial^n}{\partial \phi^n} am^{-1}(\phi|k) = \begin{cases} am^{-1}(\phi|k) \\ \dfrac{2\,\Gamma\left(n+\frac{1}{2}\right)\left(\displaystyle\sum_{\_k1=0}^{n-1} \frac{(-1)^{\_k1}\left(1-k^2\sin(\phi)^2\right)^{-\frac{1}{2}-\_k1}\left(\frac{\partial^{n-1}}{\partial \phi^{n-1}}\left(\left(1\right.\right.\right.}{\_k1!\,(2\,\_k1+1)\,(n-\_k1-1)!}\right.}{\sqrt{\pi}} \end{cases}$$

$$\frac{\partial}{\partial k} am^{-1}(\phi|k) = 2\,k\left(-\frac{Z(am^{-1}(\phi|k), k) + \frac{\boldsymbol{E}(k)\,am^{-1}(\phi|k)}{\boldsymbol{K}(k)}}{2\,k^2\,(k^2-1)} - \frac{am^{-1}(\phi|k)}{2\,k^2} + \frac{\sin(2\,\phi)}{4\,(k^2-1)\sqrt{1-k^2\sin(\phi)^2}}\right)$$

DE

$f(\phi) = am^{-1}(\phi \mid k)$	$\left(\frac{d^2}{d\phi^2} f(\phi)\right)^2 = (k^2 - 1)\left(\frac{d}{d\phi} f(\phi)\right)^6 + (-k^2 + 2)\left(\frac{d}{d\phi} f(\phi)\right)^4 - \left(\frac{d}{d\phi} f(\phi)\right)^2$

Problem
Integrate the equations of motion for a particle of mass *m* moving in a field whose potential energy is $U = A\,|x|^n$.

Solution

restart;
with(*Physics*) :

In Cartesian coordinates
Assume($x :: real$)

$$\{x::real\} \tag{364}$$

the Energy $E = T + U$ is given by
CompactDisplay($x(t)$)

$$x(t) \text{ will now be displayed as } x \tag{365}$$

$E = \frac{1}{2} m \cdot diff(x(t), t)^2 + U(x(t))$

$$E = \frac{m\,\dot{x}^2}{2} + U(x) \tag{366}$$

Introducing $U = A\,|x|^n$, we have the following 1st order differential equation for $x(t)$
Assume($n > 0, A > 0, m > 0$)

$$\{n::(0, \infty)\}, \{A::(0, \infty)\}, \{m::(0, \infty)\} \tag{367}$$

subs$\left(U(x(t)) = A\,|x(t)|^n, \textbf{(366)}\right)$

$$E = \frac{m\,\dot{x}^2}{2} + A\,|x|^n \tag{368}$$

Placing the origin of the system of reference such that $x(0) = 0$,
dsolve([**(368)**, $x(0) = 0$])

$$x = RootOf\left(t - \left(\int_0^{\_Z} \frac{m}{\sqrt{-2\,m\,\left(A\,|\_a|^n - E\right)}}\,\mathrm{d}\_a\right)\right), x = RootOf\left(t - \left(\int_0^{\_Z} -\frac{m}{\sqrt{-2\,m\,\left(A\,|\_a|^n - E\right)}}\,\mathrm{d}\_a\right)\right) \quad \textbf{(369)}$$

Take the solution for which $t > 0$

DEtools:-*remove_RootOf*(**(369)**[1])

$$t - \left(\int_0^{x} \frac{m}{\sqrt{-2\,m\,\left(A\,|\_a|^n - E\right)}}\,\mathrm{d}\_a\right) = 0 \quad \textbf{(370)}$$

The constant energy E can be replaced by its value at the return point $E = U = |x_0|^n \equiv x_0^{\ n}$

Assume$(x_0 > 0)$

$$\{x_0{::}(0, \infty)\} \quad \textbf{(371)}$$

simplify(*subs*($E = x_0^{\ n}$, **(370)**))

$$t - \frac{\sqrt{m}\,\sqrt{2}\left(\int_0^{x} \frac{1}{\sqrt{-A\,|\_a|^n + x_0^{\ n}}}\,\mathrm{d}\_a\right)}{2} = 0 \quad \textbf{(372)}$$

This integral can be computed exactly in terms of the hypergeometric function ${}_2F_1$, and the solution $x(t)$ is expressed in implicit form as $t \equiv t(x)$

value(**(372)**)

$$t - \frac{\sqrt{m}\,\sqrt{2}\left(\begin{cases} x_0^{-\frac{n}{2}}\,x\;{}_2F_1\left(\frac{1}{2}, \frac{1}{n}; \frac{1+n}{n}; x_0^{\ -n}\,A\,(-x)^n\right) & x \le 0 \\ x_0^{-\frac{n}{2}}\,x\;{}_2F_1\left(\frac{1}{2}, \frac{1}{n}; \frac{1+n}{n}; x^n\,x_0^{\ -n}\,A\right) & 0 < x \end{cases}\right)}{2} = 0 \quad \textbf{(373)}$$

Restricting this solution to the region $x > 0$,

(373) assuming $x(t) > 0$

$$t - \frac{\sqrt{m}\,\sqrt{2}\,x_0^{-\frac{n}{2}}\,x\;{}_2F_1\left(\frac{1}{2}, \frac{1}{n}; \frac{1+n}{n}; x^n\,x_0^{\ -n}\,A\right)}{2} = 0 \quad \textbf{(374)}$$

To see several sections with information on the hypergeometric function ${}_2F_1$, you can use the

FunctionAdvisor

FunctionAdvisor(hypergeom)

hypergeom

describe

hypergeom = *generalized hypergeometric function*

definition

$${}_2F_1(a, b; c; z) = \sum_{\_k1=0}^{\infty} \frac{(a)_{\_k1} \, (b)_{\_k1} \, z^{\_k1}}{\_k1! \, (c)_{\_k1}}$$

$$\left(a::\mathbb{Z}^{(0,-\,)} \land c \neq a + 1\right) \lor |z| < 1 \lor \left(|z| = 1 \land 0 < -\Re(-c + a + b)\right) \lor \left(|z| = 1 \land z \neq 1 \land -\Re(-c + a + b) \in (-1, 0]\right)$$

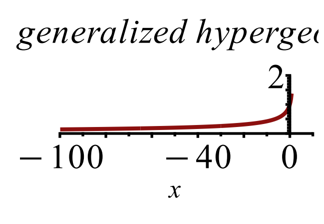
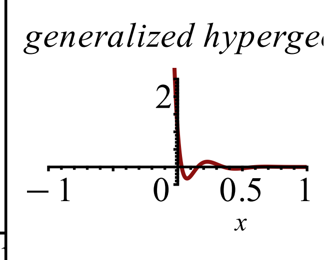
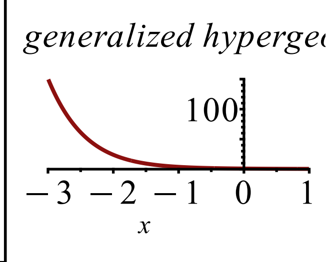
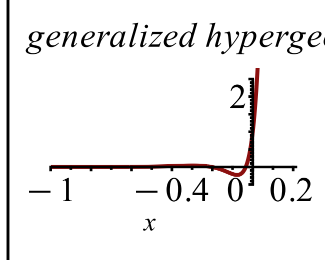
classify function

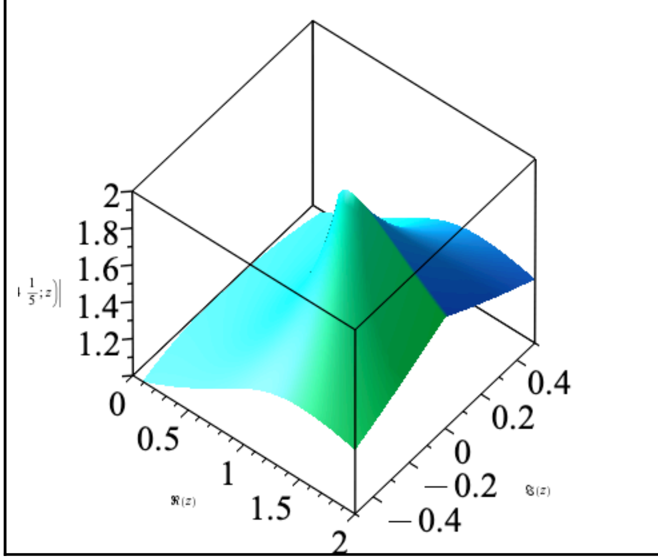
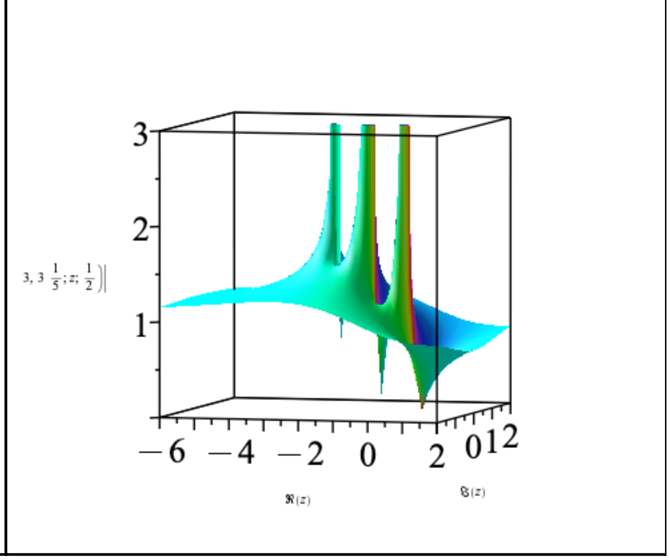
hypergeometric

periodicity

${}_2F_1(a, b; c; z)$	No periodicity

plot

			
$_2F_1\left(4\,\frac{1}{3}, 9\,\frac{1}{16}; 14\right.$	$_2F_1(5, -10; 1; x)$	$_2F_1(1, -10; 10; x$	$_2F_1(5, 10; 1; x)$

$_2F_1\left(4\,\frac{1}{3}, 9\,\frac{1}{16}; 14\,\frac{1}{5}; z\right)$	$_2F_1\left(-3, 3\,\frac{1}{5}; z; \frac{1}{2}\right)$
	

singularities

$_2F_1(a, b; c; z)$	$z \in (1, \infty)$

branch points

$_2F_1(a, b; c; z)$	$z \in [1, \infty + \infty\,\mathrm{I}]$

branch cuts

$_2F_1(a, b; c; z)$	$z \in (1, \infty)$

special values

$_2F_1(a, b; c; 0) = 1$	
$_2F_1(a, b; c; z) = b\,\Phi(z, 1, b)$	$a = 1 \wedge c = 1 + b$
$_2F_1(a, b; c; z) = T_a(1 - 2z)$	$a + b = 0 \wedge c = \frac{1}{2}$
$_2F_1(a, b; c; z) = -\frac{\ln(-z + 1)}{z}$	$a = 1 \wedge b = 1 \wedge c = 2 \wedge z \neq 0$
$_2F_1(a, b; c; z) = \frac{\operatorname{arctanh}(\sqrt{z})}{\sqrt{z}}$	$a = \frac{1}{2} \wedge b = 1 \wedge c = \frac{3}{2}$
$_2F_1(a, b; c; z) = \frac{2\,\boldsymbol{K}(\sqrt{z})}{\pi}$	$a = \frac{1}{2} \wedge b = \frac{1}{2} \wedge c = 1$
$_2F_1(a, b; c; z) = \frac{2\,\boldsymbol{E}(\sqrt{z})}{\pi}$	$a = -\frac{1}{2} \wedge b = \frac{1}{2} \wedge c = 1$
$_2F_1(a, b; c; z) = \frac{\operatorname{arccsc}\left(\frac{1}{\sqrt{z}}\right)}{\sqrt{z}}$	$a = \frac{1}{2} \wedge b = \frac{1}{2} \wedge c = \frac{3}{2}$
$_2F_1(a, b; c; z) = \frac{\arcsin(\sqrt{z})}{\sqrt{z}}$	$a = \frac{1}{2} \wedge b = \frac{1}{2} \wedge c = \frac{3}{2}$
$\arctan(\sqrt{-z})$	$a = \frac{1}{2} \wedge b = 1 \wedge c$

identities

2a2b	${}_2F_1(a, b; c; z) = {}_2F_1\left(2a, 2b; \frac{1}{2} + a + b; \frac{1}{2} - \frac{\sqrt{1-z}}{2}\right)$	$z \neq 1 \wedge \frac{1}{2} + a -$ $- c =$
lower 1/2	${}_2F_1(1, 1; 2; z) = \left({}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -\frac{z^2(z-2)^2}{4(-1+z)^2}\right) z(z-2)\right) \Big/ \left((1 - z) \sqrt{\frac{(z^2-2z+2)^2}{(-1+z)^2}} + z^2 - 2\right)$	$\Re(z) < 1$
lower a	${}_2F_1(a, b; c; z) = \frac{1}{(a-1)(-1+z)} \left(((a-b-1)z - 2a + c + 2)\, {}_2F_1(b, a-1; c; z) + (-c + a - 1)\, {}_2F_1(\right.$ $\left. a-2; c; z)\right)$	$a \neq 1 \wedge z \neq 1$
lower c	${}_2F_1(a, b; c; z) = \left(\left(((-2z+1)c - 2 + (a + b + 3)z)\, {}_2F_1(a, b; c-1; z) + {}_2F_1(\right.\right.$ $\left.\left. b; c-2; z)(-1+z)(c-2)\right)(-c + 1)\right) \Big/ ((-c+1+a)(-c+1+b)$	
quadratic 1	${}_2F_1(a, b; c; z) = \frac{1}{\sqrt{1-z}} \left({}_2F_1\left(2a\right.\right.$	$z \neq 1 \wedge$ $-\frac{1}{2}$

sum form

$$ {}_2F_1(a, b; c; z) = \sum_{\_k1=0}^{\infty} \frac{(a)_{\_k1}\,(b)_{\_k1}\,z^{\_k1}}{\_k1!\,(c)_{\_k1}} \quad \Bigg| \quad (a::\mathbb{Z}^{(0,-)} \wedge c \neq a+1) \vee |z| < 1 \vee (|z| = 1 \wedge 0 < -\Re(-c+a+b)) \vee (|z| = 1 \wedge \neq 1 \wedge -\Re(-c+a+b) \in (-1, 0]) $$

series

$$ series\left({}_2F_1(a, b; c; z), z, 4\right) = 1 + \frac{a\,b}{c}\,z + \frac{1}{2}\,\frac{a\,b\,(a+1)\,(b+1)}{c\,(c+1)}\,z^2 + \frac{1}{6}\,\frac{a\,b\,(a+1)\,(b+1)\,(a+2)\,(b+2)}{c\,(c+1)\,(c+2)}\,z^3 + \mathrm{O}\left(z^4\right) $$

integral form

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)\left(\int_0^1 \frac{\_t1^{b-1}}{(-\_t1+1)^{-c+b+1}\,(-\_t1\,z+1)^{a}}\,\mathrm{d}\_t1\right)}{\Gamma(b)\,\Gamma(c-b)} \quad \Bigg| \quad 0 < \Re(b) \wedge \Re(b) < \Re(c) $$

differentiation rule

$$ \frac{\partial}{\partial z}\,{}_2F_1(a, b; c; z) = \frac{a\,b\,{}_2F_1(a+1, b+1; c+1; z)}{c} $$

$$ \frac{\partial^n}{\partial z^n}\,{}_2F_1(a, b; c; z) = \frac{(a)_n\,(b)_n\,{}_2F_1(a+n, b+n; c+n; z)}{(c)_n} $$

DE

$f(z) = {}_2F_1(a, b; c; z)$

$$\frac{d^2}{dz^2} f(z) = \frac{((-a-b-1)z+c)\left(\frac{d}{dz} f(z)\right)}{z(z-1)} - \frac{f(z)\, a\, b}{z(z-1)}$$

Reduced mass

The two-body problem

Problem

Show that by placing the origin of the reference system at the center of mass $\vec{R} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$, the problem of two particles that interact with each other can be reduced to the problem of a single particle of mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$, herein called ***reduced mass,*** in an external field $U(\|\vec{r}_1 - \vec{r}_2\|)$.

Solution

restart;
with(*Physics*:-*Vectors*) :

In what follows $\vec{R}$ denotes the position of the center of mass while $\vec{\mathbb{R}}$ (from the palette for *Open Face*) denotes the relative position $\vec{r}_1 - \vec{r}_2$.

CompactDisplay((ℝ_, r_)(t))

$\vec{\mathbb{R}}(t)$ *will now be displayed as* $\vec{\mathbb{R}}$

$\vec{r}(t)$ *will now be displayed as* $\vec{r}$ **(375)**

The Lagrangian describing a closed system of two particles of masses m_1 and m_2 that interact with each other is given by

$L = \frac{1}{2} m[1]\, diff(r\_[1](t), t)^2 + \frac{1}{2} m[2]\, diff(r\_[2](t), t)^2 - U(Norm((r\_[1] - r\_[2])(t)))$

$$L = \frac{m_1 \dot{\vec{r}}_1^{\,2}}{2} + \frac{m_2 \dot{\vec{r}}_2^{\,2}}{2} - U(\|\vec{r}_1 - \vec{r}_2\|) \qquad \textbf{(376)}$$

Introduce the relative position vector $\vec{\mathbb{R}}$ as the dependency for the potential energy $U(\|\vec{r}_1 - \vec{r}_2\|)$

$r\_[1](t) - r\_[2](t) = \mathbb{R}\_(t)$

$$\vec{r}_1 - \vec{r}_2 = \vec{\mathbb{R}} \qquad \textbf{(377)}$$

subs(**(377)**, **(376)**)

$$L = \frac{m_1 \dot{\vec{r}}_1^{\,2}}{2} + \frac{m_2 \dot{\vec{r}}_2^{\,2}}{2} - U(\|\vec{\mathbb{R}}\|) \qquad \textbf{(378)}$$

and take the origin of the reference system at the center of mass $\vec{R}$

$\frac{\sum_{i=1}^{2} m_i \vec{r}_i(t)}{\sum_{i=1}^{2} m_i} = 0 : \ value(\%)$

$$\frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = 0 \qquad \textbf{(379)}$$

Now we have a system of two equations, **(377)** and **(379)**, that can be solved for $\vec{r}_1$ and $\vec{r}_2$

solve([**(377)**, **(379)**], $\{r\_[1], r\_[2]\}(t)$)

$$\left\{\vec{r}_1 = \frac{\vec{\mathbb{R}}\, m_2}{m_1 + m_2}, \vec{r}_2 = -\frac{m_1 \vec{\mathbb{R}}}{m_1 + m_2}\right\} \qquad \textbf{(380)}$$

The Lagrangian **(378)** can now be written in terms of $\vec{R}$

eval(**(378)**, **(380)**)

$$L = \frac{m_1 \dot{\vec{\mathbb{R}}}^2 m_2^2}{2\,(m_1 + m_2)^2} + \frac{m_2 m_1^2 \dot{\vec{\mathbb{R}}}^2}{2\,(m_1 + m_2)^2} - U(\|\vec{\mathbb{R}}\|) \qquad \textbf{(381)}$$

Expand the vector formal power $\dot{\vec{\mathbb{R}}}^2$ to express it in terms of the norm $\|\dot{\vec{\mathbb{R}}}\|^2$

expand(**(381)**)

$$L = \frac{m_1 m_2^2 \|\dot{\vec{\mathbb{R}}}\|^2}{2\,(m_1 + m_2)^2} + \frac{m_2 m_1^2 \|\dot{\vec{\mathbb{R}}}\|^2}{2\,(m_1 + m_2)^2} - U(\|\vec{\mathbb{R}}\|) \qquad \textbf{(382)}$$

Collect the *Norm* $\|\dot{\vec{R}}\|$

collect(**(382)**, *Norm*, *simplify*)

$$L = \frac{m_1 m_2 \|\dot{\vec{\mathbb{R}}}\|^2}{2 m_1 + 2 m_2} - U(\|\vec{\mathbb{R}}\|) \qquad \textbf{(383)}$$

Introduce the reduced mass

$$\frac{m[1] \cdot m[2]}{m[1] + m[2]} = \mu$$

$$\frac{m_1 m_2}{m_1 + m_2} = \mu \tag{384}$$

Simplifying the Lagrangian **(383)** taking this relation into account

simplify(**(383)**, {**(384)**})

$$L = \frac{\|\dot{\vec{\mathbb{R}}}\|^2 \mu}{2} - U(\|\vec{\mathbb{R}}\|) \tag{385}$$

This is the Lagrangian of one single particle of mass μ moving in an external field, $U(\|\vec{\mathbb{R}}\|)$, that depends only on the distance to the origin.

A many-body problem

Problem

A system consists of one particle of mass M and n particles of equal masses m.

a) Show, in steps, that eliminating the motion of the center of mass reduces the problem to one involving only n particles.

b) Show that when $n = 1$ the result of **a)** becomes the result obtained for the previous *two-body* problem, equation **(385)** $\equiv L = \frac{1}{2}\mu\,\|\dot{\vec{\mathbb{R}}}\|^2 - U$.

Solution

restart;
with(*Physics*:-*Vectors*) :

a) Let $\vec{r}_M$ represent the position vector of the particle of mass M and $\vec{r}_a$ those of the particles of mass m.

The Lagrangian $L = T - U$ is given by

CompactDisplay$((r_{M\_}, r\_, \mathbb{R}\_)(t))$

$$\vec{r}_M(t) \text{ will now be displayed as } \vec{r}_M$$

$$\vec{r}(t) \text{ will now be displayed as } \vec{r}$$

$$\vec{\mathbb{R}}(t) \text{ will now be displayed as } \vec{\mathbb{R}} \tag{386}$$

Physics:-*Assume*$(m > 0, M > 0)$

$$\{m::(0, \infty)\}, \{M::(0, \infty)\} \tag{387}$$

$L = \frac{1}{2} M \cdot diff(r_{M\_}(t), t)^2 + \frac{1}{2}\, Sum\left(m \cdot diff(r\_[a](t), t)^2, a = 1..n\right) - U$

$$L = \frac{M \dot{\vec{r}}_M^{\,2}}{2} + \frac{\left(\sum_{a=1}^{n} m \dot{\vec{r}}_a^{\,2}\right)}{2} - U \tag{388}$$

Expand the formal powers of vectors to express them in terms of their *Norm*
expand(**(388)**)

$$L = \frac{M \left\| \dot{\vec{r}}_M \right\|^2}{2} + \frac{m \left(\sum_{a=1}^{n} \left\| \dot{\vec{r}}_a \right\|^2 \right)}{2} - U \tag{389}$$

As done in the two-body problem, introduce the relative vector positions of the *n* particles with respect to the particle of mass *M*
$\mathbb{R}\_[a](t) = r\_[a](t) - r_{M\_}(t)$

$$\vec{\mathbb{R}}_a = \vec{r}_a - \vec{r}_M \tag{390}$$

and place the origin of the reference system at the center of mass,
$\frac{\left(M \cdot r_{M\_}(t) + Sum(m \cdot r\_[a](t), a = 1..n)\right)}{M + Sum(m, a = 1..n)} = 0$

$$\frac{M \vec{r}_M + \left(\sum_{a=1}^{n} m \vec{r}_a \right)}{M + \left(\sum_{a=1}^{n} m \right)} = 0 \tag{391}$$

expand(**(391)**)

$$\frac{M \vec{r}_M}{m\, n + M} + \frac{m \left(\sum_{a=1}^{n} \vec{r}_a \right)}{m\, n + M} = 0 \tag{392}$$

Using this equation and **(390)**$\equiv \vec{r}_a = \vec{\mathbb{R}}_a - \vec{r}_M$ is sufficient to eliminate $\vec{r}_M$ and $\vec{r}_a$ from the Lagrangian **(389)**. In steps, eliminating $\vec{r}_a$ from the equation for the center of mass
simplify(**(392)**, {**(390)**}, {$r\_[a](t)$})

$$\frac{M \vec{r}_M + m \left(\sum_{a=1}^{n} \left(\vec{\mathbb{R}}_a + \vec{r}_M \right) \right)}{m\, n + M} = 0 \tag{393}$$

expand(**(393)**)

$$\frac{M \vec{r}_M}{m\, n + M} + \frac{m \left(\sum_{a=1}^{n} \vec{\mathbb{R}}_a \right)}{m\, n + M} + \frac{m\, \vec{r}_M\, n}{m\, n + M} = 0 \tag{394}$$

To eliminate $\vec{r}_M$, use
simplify(*isolate*(**(394)**, $r_{M\_}(t)$))

$$\vec{r}_M = -\frac{m \left(\sum_{a=1}^{n} \vec{\mathbb{R}}_a \right)}{m\, n + M} \tag{395}$$

Likewise, to eliminate $\vec{r}_a$ use

$isolate((\mathbf{390}), r\_[a](t))$

$$\vec{r}_a = \vec{\mathbb{R}}_a + \vec{r}_M \tag{396}$$

Substitute, *sequentially*, these two equations into the Lagrangian **(389)**

$subs((\mathbf{396}), (\mathbf{395}), (\mathbf{389}))$

$$L = \frac{M \left\| \left(-\frac{m \left(\sum_{a=1}^{n} \vec{\mathbb{R}}_a \right)}{m\, n + M} \right)_t \right\|^2}{2} + \frac{m \left(\sum_{a=1}^{n} \left\| \left(\vec{\mathbb{R}}_a - \frac{m \left(\sum_{a=1}^{n} \vec{\mathbb{R}}_a \right)}{m\, n + M} \right)_t \right\|^2 \right)}{2} - U \tag{397}$$

To verify by eye each step for correctness, one can manipulate this expression surgically using subsindets.

First expand only the *Norms*

$subsindets((\mathbf{397}), specfunc(Norm), expand)$

$$L = \frac{M\, m^2 \left\| \sum_{a=1}^{n} \dot{\vec{\mathbb{R}}}_a \right\|^2}{2\, (m\, n + M)^2} + \frac{m \left(\sum_{a=1}^{n} \left(\left\| \dot{\vec{\mathbb{R}}}_a \right\|^2 - \frac{2\, m \left(\dot{\vec{\mathbb{R}}}_a \cdot \left(\sum_{a=1}^{n} \dot{\vec{\mathbb{R}}}_a \right) \right)}{m\, n + M} + \frac{m^2 \left\| \sum_{a=1}^{n} \dot{\vec{\mathbb{R}}}_a \right\|^2}{(m\, n + M)^2} \right) \right)}{2} - U \tag{398}$$

This result is correct. Next expand only the *Sums*

$subsindets((\mathbf{398}), specfunc(Sum), expand)$

$$L = \frac{M\, m^2 \left\| \sum_{a=1}^{n} \dot{\vec{\mathbb{R}}}_a \right\|^2}{2\, (m\, n + M)^2} + \frac{1}{2} \left(m \left(\frac{m^2\, n^2 \left(\sum_{a=1}^{n} \left\| \dot{\vec{\mathbb{R}}}_a \right\|^2 \right)}{(m\, n + M)^2} + \frac{2\, M\, m\, n \left(\sum_{a=1}^{n} \left\| \dot{\vec{\mathbb{R}}}_a \right\|^2 \right)}{(m\, n + M)^2} + \frac{M^2 \left(\sum_{a=1}^{n} \left\| \dot{\vec{\mathbb{R}}}_a \right\|^2 \right)}{(m\, n + M)^2} - \frac{m^2 \left\| \sum_{a=1}^{n} \dot{\vec{\mathbb{R}}}_a \right\|^2 n}{(m\, n + M)^2} - \frac{2\, M\, m \left\| \sum_{a=1}^{n} \dot{\vec{\mathbb{R}}}_a \right\|^2}{(m\, n + M)^2} \right) \right) - U \tag{399}$$

This result also verifies visually. Collect the terms polynomial in *Norm* then *Sum* and normalize the coefficients

$collect((\mathbf{399}), [Norm, Sum], normal)$

$$L = -\frac{m^2 \left\| \sum_{a=1}^{n} \dot{\vec{\mathbb{R}}}_a \right\|^2}{2\, (m\, n + M)} + \frac{m \left(\sum_{a=1}^{n} \left\| \dot{\vec{\mathbb{R}}}_a \right\|^2 \right)}{2} - U \tag{400}$$

This Lagrangian involves only the n relative position vectors $\vec{\mathbb{R}}_a$, achieving the reduction of the original problem of $n + 1$ particles to a problem of only n particles.

---

b) To obtain the result **(385)** $\equiv L = \frac{1}{2} \left\| \dot{\vec{\mathbb{R}}} \right\|^2 \mu - U$ for the *two-body* problem, substitute $n = 1$ into

$$(400) \equiv L = -\frac{\frac{1}{2} m^2 \left\| \sum_{a=1}^{n} \left(\dot{\vec{\mathbb{R}}}_a \right) \right\|^2}{m\, n + M} + \frac{1}{2} m \left(\sum_{a=1}^{n} \left\| \dot{\vec{\mathbb{R}}}_a \right\|^2 \right) - U$$

subs(*n* = 1, **(400)**)

$$L = -\frac{m^2 \left\| \sum_{a=1}^{1} \dot{\vec{\mathbb{R}}}_a \right\|^2}{2\,(M+m)} + \frac{m \left(\sum_{a=1}^{1} \left\| \dot{\vec{\mathbb{R}}}_a \right\|^2 \right)}{2} - U \qquad \textbf{(401)}$$

Expand only the *Sums* and collect the *Norm*

subsindets(**(401)**, *specfunc*(*Sum*), *expand*)

$$L = -\frac{m^2 \left\| \dot{\vec{\mathbb{R}}}_1 \right\|^2}{2\,(M+m)} + \frac{m \left\| \dot{\vec{\mathbb{R}}}_1 \right\|^2}{2} - U \qquad \textbf{(402)}$$

collect(**(402)**, *Norm*, *normal*)

$$L = \frac{M\, m \left\| \dot{\vec{\mathbb{R}}}_1 \right\|^2}{2\,(M+m)} - U \qquad \textbf{(403)}$$

Comparing with the definition of *reduced mass* **(384)**

(384)

$$\frac{m_1\, m_2}{m_1 + m_2} = \mu \qquad \textbf{(404)}$$

we see the substitutions to transform **(403)** into **(384)** are

M = *m*[1], *m* = *m*[2], *r*_[1] = *R*_

$$M = m_1,\ m = m_2,\ \vec{r}_1 = \vec{R} \qquad \textbf{(405)}$$

Substitute them, *simultaneously* (enclose the sequence of equations into a list), then simplify taking $\frac{m_1\, m_2}{m_1 + m_2} = \mu$ into account

simplify(*subs*([**(405)**], **(403)**), {**(404)**})

$$L = \frac{\left\| \dot{\vec{\mathbb{R}}}_1 \right\|^2 \mu}{2} - U \qquad \textbf{(406)}$$

This is the same as **(385)**, the reduced Lagrangian for the *two-body* problem.

Motion in a central field

A one-body problem in a central field, is about the motion of a single particle of mass *m* in a field where the potential energy, and so the magnitude of the force, depends only on the distance between the particle and a fixed point, frequently chosen as the origin of the reference system. As seen above, the two-body problem, is reducible to a one-body problem in a central field.

Problem

The angular momentum $\vec{L} = \vec{r} \times \vec{p}$ of a particle that moves in a central field is conserved, so $\vec{r}$ evolves in time on a fixed plane perpendicular to the constant $\vec{L}$. Show that the surface swept per second by the position vector $\vec{r}$ is constant (Kepler's second law).

Solution

restart;
with(*Physics*:-*Vectors*) :
CompactDisplay((*r_*, *p_*, *L_*, rho, phi, *z*, *_rho*, *_phi*)(*t*))

$$\vec{r}(t) \text{ will now be displayed as } \vec{r}$$

$$\vec{p}(t) \text{ will now be displayed as } \vec{p}$$

$$\vec{L}(t) \text{ will now be displayed as } \vec{L}$$

$$\rho(t) \text{ will now be displayed as } \rho$$

$$\phi(t) \text{ will now be displayed as } \phi$$

$$z(t) \text{ will now be displayed as } z$$

$$\hat{\rho}(t) \text{ will now be displayed as } \hat{\rho}$$

$$\hat{\phi}(t) \text{ will now be displayed as } \hat{\phi} \quad \textbf{(407)}$$

The angular momentum $\vec{L}$ of a particle in a central field
*L_* = *r_*(*t*) × *p_*(*t*)

$$\vec{L} = \vec{r} \times \vec{p} \quad \textbf{(408)}$$

subs(*p_*(*t*) = *m*·*diff*(*r_*(*t*), *t*), **(408)**)

$$\vec{L} = \vec{r} \times \left(m\,\dot{\vec{r}}\right) \quad \textbf{(409)}$$

From **(25)**, the position vector in cylindrical coordinates is given by
(25)

$$\vec{r} = z\,\hat{k} + \hat{\rho}\,\rho \quad \textbf{(410)}$$

So the *z* component of $\vec{L}$ is given by
eval(**(409)**, **(25)**)

$$\vec{L} = m\left(-z\,\dot{\phi}\,\rho\,\hat{\rho} + \left(z\,\dot{\rho} - \rho\,\dot{z}\right)\hat{\phi} + \rho^2\,\dot{\phi}\,\hat{k}\right) \quad \textbf{(411)}$$

(411) . *_k*

$$\vec{L} \cdot \hat{k} = \dot{\phi}\,\rho^2\,m \quad \textbf{(412)}$$

In the subsection on Cyclic coordinates, in equation **(339)** $\equiv \frac{\partial}{\partial\dot{\phi}} L(t) = \rho^2\,\dot{\phi}\,m$, it is shown that this quantity (as in the above) is conserved. Now, the area of an infinitesimal triangle (angle $d\phi$) on the (x, y) plane, centered at the origin of the central field, is $df = \frac{\rho \cdot (\rho\, d\phi)}{2}$. So **(412)** can be re-expressed as

$$L_z = \frac{m \cdot diff(f(t), t)}{2}$$

$$L_z = \frac{m\,\dot{f}(t)}{2} \tag{413}$$

Since by construction $f(t)$ is a *continuous* function, integrate both sides with that condition
$map(\%int, \mathbf{(413)}, t = 0..1, continuous)$

$$\int_0^1 L_z \, dt \Big|_{continuous} = \int_0^1 \frac{m\,\dot{f}(t)}{2} \, dt \Big|_{continuous} \tag{414}$$

At $t = 0$, $f(0) = 0$
$f(0) := 0$

$$f(0) := 0 \tag{415}$$

Evaluate now the integrals of **(414)**
$value(\mathbf{(414)})$

$$L_z = \frac{m\,f(1)}{2} \tag{416}$$

So the area swept in 1 second is constant and equal to
$isolate(\mathbf{(416)}, f(1))$

$$f(1) = \frac{2\,L_z}{m} \tag{417}$$

Problem

Starting from the constancy of the energy E and the angular momentum $\vec{L}$, compute the equations of movement and integrate them according to:

a) using differential elimination techniques to uncouple the system of equations of movement that involve both of $\rho(t)$ and $\phi(t)$

b) interactively, one step at a time, uncouple the equations of movement eliminating ϕ from the problem, resulting in an implicit solution $t \equiv t(\rho)$.

c) eliminate t from the problem to obtain an equation for $\frac{d\phi}{d\rho}$, whose solution is the trajectory as $\phi(\rho)$

Solution

restart;
with(*Physics*:-*Vectors*) :
CompactDisplay((rho, phi, z, _*rho*, _*phi*, *r*_)(t))

$\rho(t)$ *will now be displayed as* ρ

$\phi(t)$ *will now be displayed as* ϕ

$z(t)$ *will now be displayed as* z

$\hat{\rho}(t)$ *will now be displayed as* $\hat{\rho}$

$\hat{\phi}(t)$ *will now be displayed as* $\hat{\phi}$

$\vec{r}(t)$ *will now be displayed as* $\vec{r}$ **(418)**

$Assume\left(m > 0, L_z > 0\right)$

$$\{m::(0, \infty)\}, \{L_z::(0, \infty)\} \quad \textbf{(419)}$$

a) The energy of the system is given by
$E = T + U$

$$E = T + U \quad \textbf{(420)}$$

$T = \frac{1}{2} m \cdot diff(r\_(t), t)^2$

$$T = \frac{m \dot{\vec{r}}^{\,2}}{2} \quad \textbf{(421)}$$

In a central field, with the origin at the center of the field and using cylindrical coordinates (ρ, ϕ, z) the potential energy is $U(\|\vec{r}\|) = U(\rho)$, so the energy E is given by
$subs(\textbf{(421)}, U = U(\mathrm{rho}(t)), \textbf{(420)})$

$$E = \frac{m \dot{\vec{r}}^{\,2}}{2} + U(\rho) \quad \textbf{(422)}$$

Expand the symbolic power of $\vec{r}$ to express it in terms of its Norm squared
$expand(\textbf{(422)})$

$$E = \frac{m \left\|\dot{\vec{r}}\right\|^2}{2} + U(\rho) \quad \textbf{(423)}$$

From **(25)**, the position vector in cylindrical coordinates is
(25)

$$\vec{r} = z\,\hat{k} + \hat{\rho}\,\rho \quad \textbf{(424)}$$

Since the movement happens all on the (x, y) plane, discard the z component and introduce the result in the expression for E
$eval(\textbf{(423)}, subs(z(t) = 0, \textbf{(25)}))$

$$E = \frac{m\left(\dot{\rho}^2 + \dot{\phi}^2 \rho^2\right)}{2} + U(\rho) \quad \textbf{(425)}$$

In turn, the constancy of the z component of the angular momentum $\vec{L}$, shown in the previous problem as **(412)**
$subs\left(L\_ . \_k = L_z, \textbf{(412)}\right)$

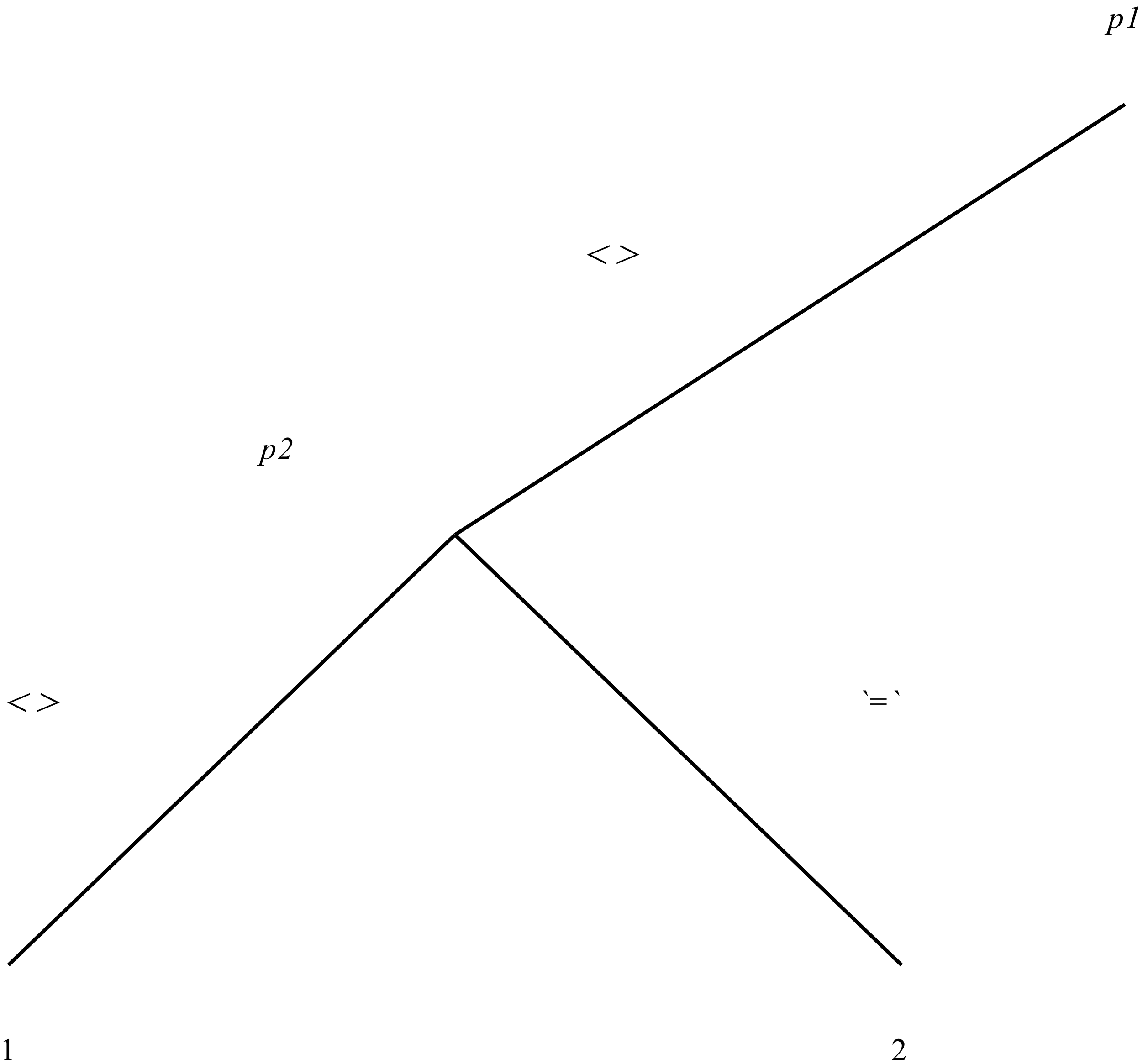
$$L_z = \dot{\phi}\,\rho^2\, m \quad \textbf{(426)}$$

Equations **(426)** and **(425)** form a system of coupled differential equations for the functions $\phi(t)$ and $\rho(t)$. Although by eye this system can be decoupled interactively with ease (see further below) it is useful to see how these coupled systems can be uncoupled in Maple using differential elimination techniques. These techniques are one of the significant advantages of computer algebra with respect to paper and pencil computations. In Maple, that uncoupling is achieved using the PDEtools:-casesplit command
$PDEtools{:}\text{-}casesplit([\textbf{(425)}, \textbf{(426)}], caseplot)$

========= *Pivots Legend* =========

$$p1 = \rho$$
$$p2 = \dot{\rho}$$

Rif Case Tree



$$\left[\dot{\phi} = \frac{L_z}{\rho^2 m}, \dot{\rho}^2 = \frac{2\rho^2 m E - 2\rho^2 m U(\rho) - L_z^2}{\rho^2 m^2}\right] \textbf{where } [\dot{\rho} \neq 0], \left[\dot{\phi} = \frac{L_z}{\rho^2 m}, U(\rho) = \frac{2\rho^2 m E - L_z^2}{2\rho^2 m}, \dot{\rho} = 0\right] \textbf{where } [\rho \neq 0] \quad \textbf{(427)}$$

In this result we see the general case in first place, with one differential equation involving only $\rho(t)$, and a singular case related to $\dot{\rho} = 0$.

---

b) The same result for the general case can be computed interactively. From **(426)**,
isolate(**(426)**, *diff*(phi(*t*), *t*))

$$\dot{\phi} = \frac{L_z}{\rho^2 m} \tag{428}$$

(428)^2

$$\dot{\phi}^2 = \frac{L_z^2}{\rho^4 m^2} \tag{429}$$

The expression of E can now be expressed only in terms of $\rho(t)$
subs(**(429)**, **(425)**)

$$E = \frac{m\left(\dot{\rho}^2 + \frac{L_z^2}{\rho^2 m^2}\right)}{2} + U(\rho) \tag{430}$$

The above is the same result shown in first place in **(427)**. Two possible solutions for $\dot{\rho}$ are
diff(rho(*t*), *t*) =~ *solve*(**(430)**, *diff*(rho(*t*), *t*))

$$\dot{\rho} = \frac{\sqrt{2\,\rho^2\, m\, E - 2\,\rho^2\, m\, U(\rho) - L_z^2}}{\rho\, m}, \dot{\rho} = -\frac{\sqrt{2\,\rho^2\, m\, E - 2\,\rho^2\, m\, U(\rho) - L_z^2}}{\rho\, m} \tag{431}$$

Taking the positive root
(431)[1]

$$\dot{\rho} = \frac{\sqrt{2\,\rho^2\, m\, E - 2\,\rho^2\, m\, U(\rho) - L_z^2}}{\rho\, m} \tag{432}$$

This is a differential equation where the variables can be separated: the right-hand side does not depend explicitly on *t*, the solution is
dsolve(**(432)**)

$$t - \left(\int^{\rho} \frac{\_a\, m}{\sqrt{2\,\_a^2\, m\, E - 2\,\_a^2\, m\, U(\_a) - L_z^2}}\, \mathrm{d}\_a\right) + c_1 = 0 \tag{433}$$

Removing the dependency of $\rho(t)$, and rewriting the Intat (an integral evaluated at a point, $\rho(t)$) as a standard integral,
useInt(*subs*(rho(*t*) = rho, **(433)**))

$$t - \left(\int \frac{\rho\, m}{\sqrt{2\,\rho^2\, m\, E - 2\,\rho^2\, m\, U(\rho) - L_z^2}}\, \mathrm{d}\rho\right) + c_1 = 0 \tag{434}$$

---

c) Since **(428)** is an expression for $\frac{d\phi}{dt}$ and **(432)** one for $\frac{d\rho}{dt}$, they can be combined into an expression for $\frac{d\phi}{d\rho}$ eliminating the independent variable *t* from the problem, resulting in an integral of the trajectory as $\phi(\rho)$

$\frac{dphi}{dt} = rhs(\textbf{(428)})$

$$\frac{d\phi}{dt} = \frac{L_z}{\rho^2 m} \qquad \textbf{(435)}$$

$\frac{drho}{dt} = rhs(\textbf{(432)})$

$$\frac{d\rho}{dt} = \frac{\sqrt{2\rho^2 m E - 2\rho^2 m U(\rho) - L_z^2}}{\rho m} \qquad \textbf{(436)}$$

$\frac{\textbf{(435)}}{\textbf{(436)}}$

$$\frac{d\phi}{d\rho} = \frac{L_z}{\rho\sqrt{2\rho^2 m E - 2\rho^2 m U(\rho) - L_z^2}} \qquad \textbf{(437)}$$

Rewriting the left-hand side as a derivative, the resulting differential equation is
$diff(\text{phi}(\text{rho}), \text{rho}) = subs(\text{rho}(t) = \text{rho}, rhs(\textbf{(437)}))$

$$\phi_\rho = \frac{L_z}{\rho\sqrt{2\rho^2 m E - 2\rho^2 m U(\rho) - L_z^2}} \qquad \textbf{(438)}$$

Solving,
$dsolve(\textbf{(438)})$

$$\phi(\rho) = \int \frac{L_z}{\rho\sqrt{2\rho^2 m E - 2\rho^2 m U(\rho) - L_z^2}}\, d\rho + c_1 \qquad \textbf{(439)}$$

Which is already the desired result $\phi(\rho)$. With some manipulations the square root can be rewritten as frequently shown in textbooks. For that, get the square root
$indets(\textbf{(439)}, \text{sqrt})$

$$\left\{\frac{1}{\sqrt{2\rho^2 m E - 2\rho^2 m U(\rho) - L_z^2}}\right\} \qquad \textbf{(440)}$$

This root can be rewritten more compactly as

$$\textbf{(440)}[1] = \frac{1}{\rho\sqrt{2m(E - U(\rho)) - \frac{L_z^2}{\rho^2}}}$$

$$\frac{1}{\sqrt{2\rho^2 m E - 2\rho^2 m U(\rho) - L_z^2}} = \frac{1}{\rho\sqrt{2m(E-U(\rho)) - \frac{L_z^2}{\rho^2}}} \quad (441)$$

So the result **(439)**, as shown in textbooks, is
subs(**(441)**, **(439)**)

$$\phi(\rho) = \int \frac{L_z}{\rho^2\sqrt{2m(E-U(\rho)) - \frac{L_z^2}{\rho^2}}}\, d\rho + c_1 \quad (442)$$

Kepler's problem

Problem

Show that, when $U(\rho) = -\frac{\alpha}{\rho}$, with $\alpha > 0$, the solution **(442)** for the motion in a central field, part **c)** of the previous problem,

$$\phi(\rho) = \int \frac{L_z}{\rho^2\left(2m(E-U(\rho)) - \frac{L_z^2}{\rho^2}\right)^{\frac{1}{2}}}\, d\rho + c_1$$

becomes the equation of a conic section

$$\frac{p}{\rho} = 1 + \xi \cdot \cos(\phi)$$

where $p = \frac{L_z^2}{m\,\alpha}$, $\xi = \sqrt{1 + \frac{2\,E\,L_z^2}{m\,\alpha^2}}$.

Solution

restart;
with(*Physics*:-*Vectors*) :

$$Assume\left(E > 0, m > 0, L_z > 0, \alpha > 0, p > 0, \xi > 0\right)$$

$$\{E::(0,\infty)\}, \{m::(0,\infty)\}, \{L_z::(0,\infty)\}, \{\alpha::(0,\infty)\}, \{p::(0,\infty)\}, \{\xi::(0,\infty)\} \quad (443)$$

Introduce the form of the potential energy $U(\rho)$ and remove the dependency of $\phi(\rho)$

$$subs\left(U(\text{rho}) = -\frac{\alpha}{\rho}, \text{phi}(\text{rho}) = \text{phi}, \mathbf{(442)}\right)$$

$$\phi = \int \frac{L_z}{\rho^2 \sqrt{2\, m \left(E + \frac{\alpha}{\rho}\right) - \frac{L_z^2}{\rho^2}}}\, d\rho + c_1 \tag{444}$$

Compute the integral

value((**444**))

$$\phi = -\arctan\left(\frac{-\alpha\, m\, \rho + L_z^2}{L_z \sqrt{2\, E\, m\, \rho^2 + 2\, \alpha\, m\, \rho - L_z^2}}\right) + c_1 \tag{445}$$

The new variables indicated, parametrizing the conic section, are

$p = \frac{L_z^2}{m\, \alpha}, \xi = \mathrm{sqrt}\left(1 + \frac{2\, E \cdot L_z^2}{m\, \alpha^2}\right)$

$$p = \frac{L_z^2}{m\, \alpha}, \xi = \sqrt{1 + \frac{2\, E\, L_z^2}{m\, \alpha^2}} \tag{446}$$

Solve these for any two of the variables involved in order to remove them from (**445**)

solve({(**446**)}, {m, alpha})

Warning, solve may be ignoring assumptions on the input variables.

$$\left\{\alpha = \frac{2\, E\, p}{\xi^2 - 1}, m = \frac{L_z^2 \left(\xi^2 - 1\right)}{2\, E\, p^2}\right\} \tag{447}$$

subs((**447**), (**445**))

$$\phi = -\arctan\left(\frac{-\frac{L_z^2\, \rho}{p} + L_z^2}{L_z \sqrt{\frac{L_z^2 \left(\xi^2 - 1\right) \rho^2}{p^2} + \frac{2\, L_z^2\, \rho}{p} - L_z^2}}\right) + c_1 \tag{448}$$

simplify((**448**))

$$\phi = -\arctan\left(\frac{p - \rho}{\sqrt{\left(\xi^2 - 1\right) \rho^2 + 2\, \rho\, p - p^2}}\right) + c_1 \tag{449}$$

At this point, for manipulation purposes it is convenient to introduce a single variable representing the rate

$\frac{p}{\rho} \equiv P(\rho)$

simplify(*subs*($p = P$(rho)$\cdot$rho, (**449**)))

$$\phi = -\arctan\left(\frac{P(\rho) - 1}{\sqrt{-P(\rho)^2 + \xi^2 + 2\, P(\rho) - 1}}\right) + c_1 \tag{450}$$

Solving for $P(\rho)$,

$P(\text{rho}) = solve((\mathbf{450}), P(\text{rho}))$

Warning, solve may be ignoring assumptions on the input variables.

$$P(\rho) = \tan(-\phi + c_1)\,\xi\,\sqrt{\frac{1}{\tan(-\phi + c_1)^2 + 1}} + 1 \tag{451}$$

Adjusting the value of the integration constant $c_1 = \frac{\pi}{2}$,

$simplify\left(eval\left((\mathbf{451}), c_1 = \frac{\pi}{2}\right)\right)$ assuming $\sin(\phi) > 0$

$$P(\rho) = \cos(\phi)\,\xi + 1 \tag{452}$$

Recalling the definition of $P(\rho)$,

$subs\left(P(\rho) = \frac{p}{\rho}, (\mathbf{452})\right)$

$$\frac{p}{\rho} = \cos(\phi)\,\xi + 1 \tag{453}$$

This is the expected result, the equation of a conic section with one focus at the origin, $2\,p$ as latus rectum and ξ representing the eccentricity.

Small Oscillations

Free oscillations in one dimension

Problem

Consider the case of 1-dimensional motion where the acting force opposes the motion as a function of the position, $\vec{F} = -k\,x\,\hat{i}$. This is the case, for example, of a spring, the more you stretch it (the bigger x), the more it opposes the stretching in the opposite direction $(-\hat{i})$. Write the equation of motion as Newton's 2^{nd} law, then the Lagrangian and Lagrange equation for it, and integrate the equation for generic initial conditions

Solution

restart :
with(*Physics*:-*Vectors*) :
with(*Physics*, *LagrangeEquations*) :

Since we know the force, we can write Newton's 2^{nd} law $\vec{F} = m\,\vec{a}$ directly

CompactDisplay($x(t)$)

$$x(t)\ \textit{will now be displayed as}\ x \tag{454}$$

$F\_ := -k\,x(t)\,\_i$

$$\vec{F} := -k\,x\,\hat{i} \tag{455}$$

$m\ diff(x(t)\,\_i, t, t) = F\_$

$$m \ddot{x}\, \hat{i} = -k x\, \hat{i} \tag{456}$$

It is instructive, however, to formulate the problem deriving the equations of motion **(456)** from the Lagrangian. The potential energy associated to this force is, up to an arbitrary non-relevant constant,

$U := \frac{k\, x(t)^2}{2}$

$$U := \frac{k x^2}{2} \tag{457}$$

so that the force F can be retrieved as $-\nabla U$

$-(\%Nabla = Nabla)(U)$

$$-\nabla\left(\frac{k x^2}{2}\right) = -k x\, \hat{i} \tag{458}$$

The kinetic energy is

$T := \frac{1}{2}\, m\, diff(x(t), t)^2$

$$T := \frac{m \dot{x}^2}{2} \tag{459}$$

so the Lagrangian is given by

$L := T - U$

$$L := \frac{m \dot{x}^2}{2} - \frac{k x^2}{2} \tag{460}$$

The Lagrange equation involves only these two terms:

$\%diff(\%diff(L, diff(x(t), t)), t) = \%diff(L, x(t))$

$$\frac{\partial}{\partial t} \frac{\partial}{\partial \dot{x}}\left(\frac{m \dot{x}^2}{2} - \frac{k x^2}{2}\right) = \frac{\partial}{\partial x}\left(\frac{m \dot{x}^2}{2} - \frac{k x^2}{2}\right) \tag{461}$$

$value(\mathbf{(461)})$

$$m \ddot{x} = -k x \tag{462}$$

where we see Newton's second law **(456)** retrieved. The (Lagrange) equation of motion **(461)** can be computed directly using the LagrangeEquations command

$LagrangeEquations(L, x)$

$$m \ddot{x} + k x = 0 \tag{463}$$

The integration of this equation is straightforward. Generic initial conditions would be

$\%eval(x(t), t = t_0) = x_0,\ \%eval(diff(x(t), t), t = t_0) = v_0$

$$x\Big|_{t = t_0} = x_0,\ \dot{x}\Big|_{t = t_0} = v_0 \tag{464}$$

$dsolve([\mathbf{(456)} \cdot \_i, \mathbf{(464)}])$

$$x = \frac{\left(\cos\left(\frac{\sqrt{k}\, t_0}{\sqrt{m}}\right) \sqrt{m}\, v_0 + \sin\left(\frac{\sqrt{k}\, t_0}{\sqrt{m}}\right) \sqrt{k}\, x_0\right) \sin\left(\frac{\sqrt{k}\, t}{\sqrt{m}}\right)}{\sqrt{k}} \tag{465}$$

$$+\frac{\left(x_0\sqrt{k}\cos\left(\frac{\sqrt{k}\,t_0}{\sqrt{m}}\right)-v_0\sqrt{m}\sin\left(\frac{\sqrt{k}\,t_0}{\sqrt{m}}\right)\right)\cos\left(\frac{\sqrt{k}\,t}{\sqrt{m}}\right)}{\sqrt{k}}$$

Forced oscillations

Problem

Consider oscillations in 1 dimension of a system on which an external force $\vec{F}_{ext}(t)$ acts. For the oscillations to be small, $\vec{F}_{ext}(t)$ must produce only small displacements. The total force is $\vec{F} = -k\,x\,\hat{i} + \vec{F}_{ext}(t)$.

a) Write the equation of motion as Newton's 2^{nd} law, then write the Lagrangian and Lagrange equations.
b) Integrate the equation of motion for generic initial conditions.
c) Specialize the solution computed in **b)** for $\left\|\vec{F}_{ext}(t)\right\| = f_0 \cdot \cos(\lambda\,t+\beta)$ to obtain

$$x(t) = \frac{a\cos(\omega\,t+\alpha) - \frac{f_0\,\omega\cos(\lambda\,t+\beta)}{m\,(\lambda^2-\omega^2)} - \frac{f_0\cos(-\omega\,t+\beta)}{2\,m\,(\lambda+\omega)} + \frac{f_0\cos(\omega\,t+\beta)}{2\,(\lambda-\omega)\,m}}{\omega}$$

for some constants a, b, α, β.
d) Show that a solution for the case considered in **c)**, that is, $\left\|\vec{F}_{ext}(t)\right\| = f_0 \cdot \cos(\lambda\,t+\beta)$, can be computed manually to get

$$x(t) = a\cos(\omega\,t+\alpha) + \frac{f_0\cos(\lambda\,t+\beta)}{m\,(\omega^2-\lambda^2)}$$

for some other constants a, b, α, β.
e) Specialize the solution of item **d)** in the case of resonance, when $\lambda = \omega$, by taking limits, thus obtaining

$$x(t) = a\cos(\omega\,t+\alpha) + \frac{f_0\,t\sin(\omega\,t+\beta)}{2\,\omega\,m}$$

f) Show that the solution computed taking limits in **e)** can be compute directly by using *dsolve* and specializing the integration constants c_1 and c_2 that appear when solving the underlying differential equation.

Solution

restart :
with(*Physics*:-*Vectors*) :
with(*Physics*, *Gtaylor*, *Coefficients*, *LagrangeEquations*) :
CompactDisplay(*x*(*t*))

x(*t*) *will now be displayed as x* **(466)**

Since $\vec{F}_{ext}(t)$ produces small displacements, and assuming it derives from a potential $U_{ext}(x(t))$, it can be approximated by expanding $U_{ext}(x(t))$ in series while keeping terms up to first order in x

$U_{ext}(x(t)) = Gtaylor\left(U_{ext}(x(t)), x(t), 2\right)$

$$U_{ext}(x) = U_{ext}(0) + \mathrm{D}\left(U_{ext}\right)(0)\, x \tag{467}$$

Denoting $F_0(t) = -\mathrm{D}\left(U_{ext}\right)(0)$ and taking the Gradient

$F\_[ext](t) = subs\left(\mathrm{D}\left(U_{ext}\right)(0) = -F_0(t), -\ Gradient(rhs((\mathbf{467})))\right)$

$$\vec{F}_{ext}(t) = F_0(t)\ \hat{i} \tag{468}$$

The total force is

$F\_(t) = rhs((\mathbf{468})) - k\, x(t)\ \_i$

$$\vec{F}(t) = \hat{i}\ \left(F_0(t) - k\, x\right) \tag{469}$$

We can write Newton's 2nd law $\vec{F} = m\ \vec{a}$ as

$m\ \cdot diff(x(t)\ \_i, t, t) = rhs((\mathbf{469}))$

$$m\, \ddot{x}\ \hat{i} = \hat{i}\ \left(F_0(t) - k\, x\right) \tag{470}$$

Since there is only one component, one may prefer to write this as

$(\mathbf{470})\ .\ \_i$

$$m\, \ddot{x} = F_0(t) - k\, x \tag{471}$$

Using $Component((\mathbf{470}), 1)$ produces the same result. The Lagrangian can be formulated directly by taking **(467)** as the potential for $\vec{F}_{ext}$ and $\frac{k\, x(t)^2}{2}$ as the potential for free oscillations

$U := \frac{k\, x(t)^2}{2} + subs\left(\mathrm{D}\left(U_{ext}\right)(0) = -F_0(t), rhs((\mathbf{467}))\right)$

$$U := \frac{k\, x^2}{2} + U_{ext}(0) - F_0(t)\, x \tag{472}$$

Discard the term $U_{ext}(0)$ which can always be expressed as a total derivative with respect to the time t

$U := subs\left(U_{ext}(0) = 0, U\right)$

$$U := \frac{k\, x^2}{2} - F_0(t)\, x \tag{473}$$

The force F can be retrieved as $-\nabla U$

$-(\%Nabla = Nabla)(U)$

$$-\nabla\left(\frac{k\, x^2}{2} - F_0(t)\, x\right) = -\left(k\, x - F_0(t)\right)\ \hat{i} \tag{474}$$

The kinetic energy is

$T := \frac{1}{2}\ m\ diff(x(t), t)^2$

$$T := \frac{m\, \dot{x}^2}{2} \tag{475}$$

so the Lagrangian is given by

$L := T - U$

$$L := \frac{m \dot{x}^2}{2} - \frac{k x^2}{2} + F_0(t)\, x \quad \textbf{(476)}$$

The Lagrange equation is

%diff(*%diff*(*L*, *diff*(*x*(*t*), *t*)), *t*) = *%diff*(*L*, *x*(*t*))

$$\frac{\partial}{\partial t} \frac{\partial}{\partial \dot{x}} \left(\frac{m \dot{x}^2}{2} - \frac{k x^2}{2} + F_0(t)\, x \right) = \frac{\partial}{\partial x} \left(\frac{m \dot{x}^2}{2} - \frac{k x^2}{2} + F_0(t)\, x \right) \quad \textbf{(477)}$$

value(**(477)**)

$$m \ddot{x} = F_0(t) - k x \quad \textbf{(478)}$$

Here we see Newton's second law **(471)** retrieved; this equation of motion can also be computed using the LagrangeEquations command

LagrangeEquations(*L*, *x*)

$$-m \ddot{x} + F_0(t) - k x = 0 \quad \textbf{(479)}$$

Introducing the frequency $\omega^2 = \frac{k}{m}$

simplify$\left(\textbf{(478)}, \left\{\omega^2 = \frac{k}{m}\right\}, \{k\}\right)$

$$m \ddot{x} = -x \omega^2 m + F_0(t) \quad \textbf{(480)}$$

---

b) The integration of this equation with generic initial conditions gives

%eval(*x*(*t*), *t* = 0) = x_0, *%eval*(*diff*(*x*(*t*), *t*), *t* = 0) = v_0

$$x\Big|_{t=0} = x_0,\ \dot{x}\Big|_{t=0} = v_0 \quad \textbf{(481)}$$

dsolve([**(480)**, **(481)**])

$$x = \frac{1}{m \omega} \left(\cos(\omega t)\, x_0\, m\, \omega + \sin(\omega t)\, v_0\, m + \left(\int_0^t \cos(\omega\, \_z1)\, F_0(\_z1)\, \mathrm{d}\_z1 \right) \sin(\omega t) - \left(\int_0^t \sin(\omega\, \_z1)\, F_0(\_z1)\, \mathrm{d}\_z1 \right) \cos(\omega t) \right) \quad \textbf{(482)}$$

combine(**(482)**)

$$x = \frac{-\left(\int_0^t F_0(\_z1) \sin(\omega\, \_z1 - \omega t)\, \mathrm{d}\_z1 \right) + \cos(\omega t)\, x_0\, m\, \omega + \sin(\omega t)\, v_0\, m}{\omega m} \quad \textbf{(483)}$$

The last two terms of the numerator can be rewritten as

$\cos(\omega t)\, x_0\, \omega + \sin(\omega t)\, v_0 = a \cdot \cos(\omega \cdot t + \text{alpha})$

$$\cos(\omega t)\, \omega x_0 + \sin(\omega t)\, v_0 = a \cos(\omega t + \alpha) \quad \textbf{(484)}$$

Expanding the right-hand side,

lhs(**(484)**) = *expand*(*rhs*(**(484)**))

$$\cos(\omega t)\,\omega x_0 + \sin(\omega t)\,v_0 = a\cos(\omega t)\cos(\alpha) - a\sin(\omega t)\sin(\alpha) \quad \textbf{(485)}$$

the relation between x_0, v_0 and a, α is

PDEtools:-*Solve*(**(485)**, $\{x_0, v_0\}$, *independentof* = t)

$$\left\{v_0 = -a\sin(\alpha),\ x_0 = \frac{a\cos(\alpha)}{\omega}\right\} \quad \textbf{(486)}$$

Hence the solution **(483)** can be rewritten as

simplify(**(483)**, {**(484)**})

$$x = \frac{a\,m\cos(\omega t + \alpha) - \left(\int_0^t F_0(\_z1)\sin(\omega(\_z1 - t))\,\mathrm{d}\_z1\right)}{m\,\omega} \quad \textbf{(487)}$$

---

c) The external force is also oscillatory, of the form $F_0(t) = f_0 \cdot \cos(\lambda t + \beta)$. Transform this form of $F_0(t)$ into a *mapping* using unapply in order to use it in the differential equation **(480)** and within the integral above

$F_0 = unapply(f_0 \cdot \cos(\lambda t + \beta), t)$

$$F_0 = (t \mapsto f_0 \cdot \cos(\lambda \cdot t + \beta)) \quad \textbf{(488)}$$

The equation of motion becomes

eval(**(480)**, **(488)**)

$$m\,\ddot{x} = -x\,\omega^2\,m + f_0\cos(\lambda t + \beta) \quad \textbf{(489)}$$

and for the solution **(487)**

eval(**(487)**, **(488)**)

$$x = \frac{a\,m\cos(\omega t + \alpha) - \left(\int_0^t f_0\cos(\lambda\,\_z1 + \beta)\sin(\omega(\_z1 - t))\,\mathrm{d}\_z1\right)}{m\,\omega} \quad \textbf{(490)}$$

The integral can now be evaluated. However, the result is slightly messy

value(**(490)**)

$$x = \frac{1}{m\,\omega}\left(a\,m\cos(\omega t + \alpha) + \frac{1}{2(\lambda - \omega)(\lambda + \omega)}\left(f_0\left(\cos(\omega t + \beta)\lambda + \cos(\omega t + \beta)\omega - \cos(-\omega t + \beta)\lambda + \cos(-\omega t + \beta)\omega - 2\cos(\lambda t + \beta)\omega\right)\right)\right) \quad \textbf{(491)}$$

In cases like this, a better result can be computed by doing a surgical operation: evaluate the integral and only in the result collect *cos* functions simplifying their coefficients

subsindets(**(490)**, *specfunc*(*Int*), $u \to$ *collect*(*value*(u), cos, *simplify*))

$$x = \frac{a\,m\cos(\omega t + \alpha) - \frac{f_0\,\omega\cos(\lambda t + \beta)}{\lambda^2 - \omega^2} - \frac{f_0\cos(-\omega t + \beta)}{2\lambda + 2\omega} - \frac{f_0\cos(\omega t + \beta)}{-2\lambda + 2\omega}}{m\,\omega} \quad \textbf{(492)}$$

These forms of the solution, **(491)** or **(492)**, can be verified by substituting into **(489)** or by using odetest
odetest(**(492)**, **(489)**)

$$0 \tag{493}$$

---

d) A solution to the equation of motion **(489)** $\equiv m\,\ddot{x} = -x\,\omega^2\,m + f_0\cos(\lambda\,t + \beta)$ can also be computed set-by-step, instead of directly calling *dsolve* with initial conditions as done in **(482)**, by noting that it is a *linear non-homogeneous* equation. For such an equation, a general solution can be computed as the sum of the solution $x_h(t)$ of the *homogeneous part,* $m\,\ddot{x}(t) = -k\,x(t)$ plus any particular solution $x_p(t)$ of the complete *non-homogeneous* equation
$x(t) = x_h(t) + x_p(t)$

$$x = x_h(t) + x_p(t) \tag{494}$$

For $x_h(t)$, we have
$subs\left(f_0 = 0, x = x_h, \textbf{(489)}\right)$

$$m\,\ddot{x}_h(t) = -x_h(t)\,\omega^2\,m \tag{495}$$

dsolve(**(495)**)

$$x_h(t) = c_1\sin(\omega\,t) + c_2\cos(\omega\,t) \tag{496}$$

As seen in **(484)** this expression can be rewritten
$rhs(\textbf{(496)}) = a\cdot\cos(\omega\,t + \alpha)$

$$c_1\sin(\omega\,t) + c_2\cos(\omega\,t) = a\cos(\omega\,t + \alpha) \tag{497}$$

where on the right-hand side there are also two constants a and α
expand(**(497)**)

$$c_1\sin(\omega\,t) + c_2\cos(\omega\,t) = a\cos(\omega\,t)\cos(\alpha) - a\sin(\omega\,t)\sin(\alpha) \tag{498}$$

$PDEtools{:}\text{-}Solve\left(\textbf{(498)}, \{c_1, c_2\}, independentof = t\right)$

$$\{c_1 = -a\sin(\alpha), c_2 = a\cos(\alpha)\} \tag{499}$$

Therefore
subs(**(499)**, **(496)**)

$$x_h(t) = a\cos(\omega\,t)\cos(\alpha) - a\sin(\omega\,t)\sin(\alpha) \tag{500}$$

combine(**(500)**)

$$x_h(t) = a\cos(\omega\,t + \alpha) \tag{501}$$

For $x_p(t)$, *any* particular solution of **(489)** $\equiv m\,\ddot{x} = -x\,\omega^2\,m + f_0\cos(\lambda\,t + \beta)$ will suffice. Search for a solution with the same frequency λ entering $F_0(t)$
$x_p(t) = b\cdot\cos(\lambda\,t + \beta)$

$$x_p(t) = b\cos(\lambda\,t + \beta) \tag{502}$$

Insert this into the equation of movement **(489)** and solve for b
$eval(\textbf{(489)}, x(t) = rhs(\textbf{(502)}))$

$$-m\,b\,\lambda^2\cos(\lambda t+\beta) = -b\cos(\lambda t+\beta)\,\omega^2 m + f_0\cos(\lambda t+\beta) \quad \textbf{(503)}$$

PDEtools:-*Solve*(**(503)**, *b*, *independentof* = *t*)

$$b = -\frac{f_0}{m\,(\lambda^2-\omega^2)} \quad \textbf{(504)}$$

So for $x_p(t)$ we get

subs(**(504)**, **(502)**)

$$x_p(t) = -\frac{f_0\cos(\lambda t+\beta)}{m\,(\lambda^2-\omega^2)} \quad \textbf{(505)}$$

The general solution to **(489)** $\equiv m\,\ddot{x} = -x\,\omega^2 m + f_0\cos(\lambda t+\beta)$ can then be written as

subs(**(500)**, **(505)**, **(494)**)

$$x = a\cos(\omega t)\cos(\alpha) - a\sin(\omega t)\sin(\alpha) - \frac{f_0\cos(\lambda t+\beta)}{m\,(\lambda^2-\omega^2)} \quad \textbf{(506)}$$

collect(*combine*(**(506)**), cos, *simplify*)

$$x = a\cos(\omega t+\alpha) - \frac{f_0\cos(\lambda t+\beta)}{m\,(\lambda^2-\omega^2)} \quad \textbf{(507)}$$

This solution can be verified by substituting into **(489)** or by using odetest

odetest(**(507)**, **(489)**)

$$0 \quad \textbf{(508)}$$

---

e) In the case of resonance, when $\omega = \lambda$, the solution **(506)** is not valid. Still, a valid solution can be computed from **(506)** by taking limits, though a direct approach leads nowhere

$Assume\left(m > 0, \omega > 0, \lambda > 0, f_0 > 0, a :: real, b :: real\right)$

$$\{m::(0,\infty)\}, \{\omega::(0,\infty)\}, \{\lambda::(0,\infty)\}, \{f_0::(0,\infty)\}, \{a::real\}, \{b::real\} \quad \textbf{(509)}$$

limit(**(507)**, $\lambda = \omega$)

$$x = \lim_{\lambda\to\omega}\left(a\cos(\omega t+\alpha) - \frac{f_0\cos(\lambda t+\beta)}{m\,(\lambda^2-\omega^2)}\right) \quad \textbf{(510)}$$

This limit, however, can be computed by applying the L'Hopital rule. Split the right-hand side into numerator and denominator

$f, g :=$ (*numer*, *denom*)(*rhs*(**(507)**))

$$f, g := \cos(\omega t+\alpha)\,a\,\lambda^2 m - a\,m\,\omega^2\cos(\omega t+\alpha) - f_0\cos(\lambda t+\beta),\ m\,(\lambda^2-\omega^2) \quad \textbf{(511)}$$

Since the limit of both f and g when $\lambda \to \omega$ exists

limit(f, $\lambda = \omega$)

$$-f_0\cos(\omega t+\beta) \quad \textbf{(512)}$$

limit(g, $\lambda = \omega$)

$$0 \quad \textbf{(513)}$$

the limit **(510)** can be computed by taking the ratio of the derivatives of *f* and *g*

limit(*diff*(*f*, λ), λ = ω)

$$2\, a \cos(\omega t + \alpha)\, m\, \omega + f_0\, t \sin(\omega t + \beta) \quad \textbf{(514)}$$

limit(*diff*(*g*, λ), λ = ω)

$$2\, m\, \omega \quad \textbf{(515)}$$

Collect the terms in *sin* and *cos* for readability

$$collect\left(lhs(\textbf{(510)}) = \frac{\textbf{(514)}}{\textbf{(515)}}, [\sin, \cos] \right)$$

$$x = \frac{f_0\, t \sin(\omega t + \beta)}{2\, \omega\, m} + a \cos(\omega t + \alpha) \quad \textbf{(516)}$$

This solution can be verified: the differential equation in the case of resonance is

subs(λ = ω, **(489)**)

$$m\, \ddot{x} = -x\, \omega^2\, m + f_0 \cos(\omega t + \beta) \quad \textbf{(517)}$$

odetest(**(516)**, **(517)**)

$$0 \quad \textbf{(518)}$$

---

f) The solution above for λ = ω can be computed directly by applying *dsolve* to the differential equation of the resonance case **(517)**

dsolve(**(517)**)

$$x = \sin(\omega t)\, c_2 + \cos(\omega t)\, c_1 + \frac{f_0\, (2\, \omega\, t \sin(\omega t + \beta) + \cos(\omega t + \beta))}{4\, \omega^2\, m} \quad \textbf{(519)}$$

To adjust the integration constants c_1 and c_2 in the above to match the compact solution **(516)** that was computed interactively, start by expanding the difference of these two solutions

expand(**(516)** − **(519)**)

$$0 = a \cos(\omega t) \cos(\alpha) - a \sin(\omega t) \sin(\alpha) - \sin(\omega t)\, c_2 - \cos(\omega t)\, c_1 - \frac{f_0 \cos(\omega t) \cos(\beta)}{4\, \omega^2\, m} + \frac{f_0 \sin(\omega t) \sin(\beta)}{4\, \omega^2\, m} \quad \textbf{(520)}$$

Now solve this equation for $\{c_1, c_2\}$ independent of *t*

PDEtools:-*Solve*(**(520)**, $\{c_1, c_2\}$, *independentof* = *t*)

$$\left\{ c_1 = \frac{4\, a \cos(\alpha)\, \omega^2\, m - \cos(\beta)\, f_0}{4\, \omega^2\, m},\ c_2 = -\frac{4\, a \sin(\alpha)\, \omega^2\, m - \sin(\beta)\, f_0}{4\, \omega^2\, m} \right\} \quad \textbf{(521)}$$

Substitute and simplify to get **(516)**

subs(**(521)**, **(519)**)

$$x = -\frac{\sin(\omega t)\left(4 a \sin(\alpha)\omega^2 m - \sin(\beta) f_0\right)}{4\omega^2 m} + \frac{\cos(\omega t)\left(4 a \cos(\alpha)\omega^2 m - \cos(\beta) f_0\right)}{4\omega^2 m} + \frac{f_0 (2\omega t \sin(\omega t + \beta) + \cos(\omega t + \beta))}{4\omega^2 m} \quad \textbf{(522)}$$

collect(*combine*(**(522)**), [sin, cos])

$$x = \frac{f_0 t \sin(\omega t + \beta)}{2\omega m} + a \cos(\omega t + \alpha) \quad \textbf{(523)}$$

Oscillations of systems with many degrees of freedom

Problem

Formulate the equations of motion for the free oscillations of a system with n degrees of freedom as

$$m_{a,c}\ddot{x}_a + k_{a,c} x_a = 0$$

where x_a represents the displacement of the a^{th} generalized coordinate q_a, the index a runs from 1 to n and there is an implicit sum over repeated indices (Einstein's convention).

Solution

restart;
with(*Physics*) : *with*(*Vectors*) :
CompactDisplay($x(t)$)

$$x(t) \text{ will now be displayed as } x \quad \textbf{(524)}$$

Denoting the generalized coordinates by q_a, the potential energy $U(q_a)$ can be expanded in series around the minimums q_{a0}. Since the movement consist only in small displacements $x_a = q_a - q_{a0}$ around q_{a0}, it is sufficient to keep terms in the expansion up to order 2, resulting in an expression analogous to $U = \frac{1}{2} k x^2$ of the 1-dimensional case:

$U = \frac{1}{2}$ *Sum*(*Sum*(k[a, b] x[a](t) x[b](t), i = 1 ..n), j = 1 ..n)

$$U = \frac{\left(\sum_{j=1}^{n}\sum_{i=1}^{n} k_{a,b} x_a x_b\right)}{2} \quad \textbf{(525)}$$

Likewise, for the kinetic energy,

$T = \frac{1}{2}$ *Sum*(*Sum*(A[a, b](q_0) *diff*(x[a](t), t) *diff*(x[b](t), t), a = 1 ..n), b = 1 ..n)

$$T = \frac{\left(\sum_{b=1}^{n} \sum_{a=1}^{n} A_{a,b}(q_0)\, \dot{x}_a \dot{x}_b\right)}{2} \tag{526}$$

where we take the $A_{a,b}$ at the minimums q_0 and denote them as $m_{i,j}$

$subs\left(A_{a,b}(q_0) = m[a,b], \textbf{(526)}\right)$

$$T = \frac{\left(\sum_{b=1}^{n} \sum_{a=1}^{n} m_{a,b}\, \dot{x}_a \dot{x}_b\right)}{2} \tag{527}$$

Both $k_{a,b}$ and $m_{a,b}$ can be split into symmetric and antisymmetric parts, with the antisymmetric parts canceling out in view of the symmetric character of $x_a\, x_b$ in U and $\dot{x}_a \dot{x}_b$ in T. Therefore, we can take $k_{a,b}$ and $m_{a,b}$ as symmetric without any loss of generality.

Before proceeding, note the similarity in notation between the three formulas **(525)** to **(527)** for T and U and tensor notation. In T and U the x_a describe *independent* displacements, so one can think of x_a as a tensor in an Euclidean *space of displacements* of generic abstract dimension *n,* with KroneckerDelta as the metric. It is then simpler to write the Lagrangian using tensor notation with a *generic* type of index (that admits and abstract *n*-dimension). For this purpose, introduce *lowercaselatin* indices from *a* to *h* to represent *generic* indices, and when necessary use KroneckerDelta as the metric.

Setup(*genericindices* = *lowercaselatin_ah*)

$$[genericindices = lowercaselatin\_ah] \tag{528}$$

Now introduce the tensors while making sure to indicate $m_{a,b}$ and $k_{a,b}$ are *symmetric* (passing x_a together is not a problem, it has only one index)

Define($x[a]$, $m[a,b]$, $k[a,b]$, *symmetric*)

Defined objects with tensor properties

$$\left\{\gamma_{\mu}, \sigma_{\mu}, \partial_{\mu}, g_{\mu,\nu}, k_{a,b}, m_{a,b}, x_a, \epsilon_{\alpha,\beta,\mu,\nu}\right\} \tag{529}$$

The Lagrangian $L = T - U$ can now be written as

$L := \frac{1}{2}\left(m[a,b]\, diff(x[a](t), t)\, diff(x[b](t), t) - k[a,b]\, x[a](t)\, x[b](t)\right)$

$$L := \frac{m_{a,b}\, \dot{x}_a \dot{x}_b}{2} - \frac{k_{a,b}\, x_a x_b}{2} \tag{530}$$

where Einstein's summation rule for repeated indices is used. Einstein's rule is taken into account by the system when differentiating, computing products and simplifying tensor indices. The simplest way to compute the Lagrange equations is to use the LagrangeEquations command

LagrangeEquations(L, x)

$$k_{a,c}\, x_a + m_{a,c}\, \ddot{x}_a = 0 \tag{531}$$

The same result can be computed via $\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{x}_c} = \frac{\partial L}{\partial x_c}$. Although not necessary, enclose the operation with forward quotes to delay its evaluation in order to see what is being computed

' *%diff*(*%diff*(L, *diff*($x[c](t)$, t)), t) = *%diff*(L, $x[c](t)$) '

$$\frac{\partial}{\partial t}\frac{dL}{d\dot{x}_c} = \frac{dL}{dx_c} \tag{532}$$

value((**532**))

$$\frac{m_{a,b}\ddot{x}_a\delta_{b,c}}{2} + \frac{m_{a,b}\ddot{x}_b\delta_{a,c}}{2} = -\frac{k_{a,b}x_a\delta_{b,c}}{2} - \frac{k_{a,b}x_b\delta_{a,c}}{2} \tag{533}$$

Simplifying tensor indices,
Simplify((**533**))

$$m_{a,c}\ddot{x}_a = -k_{a,c}x_a \tag{534}$$

which is the same as (**531**).

Rigid-body motion

A rigid body is one where (in approximation) the distances between the body's parts remain unchanged. In what follows, for simplicity, the body is considered as discrete set of particles; the formulas for a *continuous* body can be obtained from those by replacing the masses m_i of each particle by $\rho(\vec{r})\,dV$, where $\rho(\vec{r})$ is the mass density as a function of the position and dV is the volume element, whose integration represent the body's volume.

This problem is systematically treated by using two reference systems: an inertial one K, where the observer is, and another one K', rigidly attached to the body, that moves with it and thus it is typically *non-inertial*. It is customary (not necessary) to place the origin $\vec{R}(t)$ of K' at the body's center of mass.

A rigid body is thus a system with six degrees of freedom: three indicating the position $\vec{R}(t)$ of the center of mass plus three angles specifying the orientation of the axes of K' with respect to those of K.

Angular velocity

Problem
a) Show, using graphs, that the velocity $\vec{v}$ of a point P of a body, measured in an inertial reference system K, can be written as

$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}\,'$$

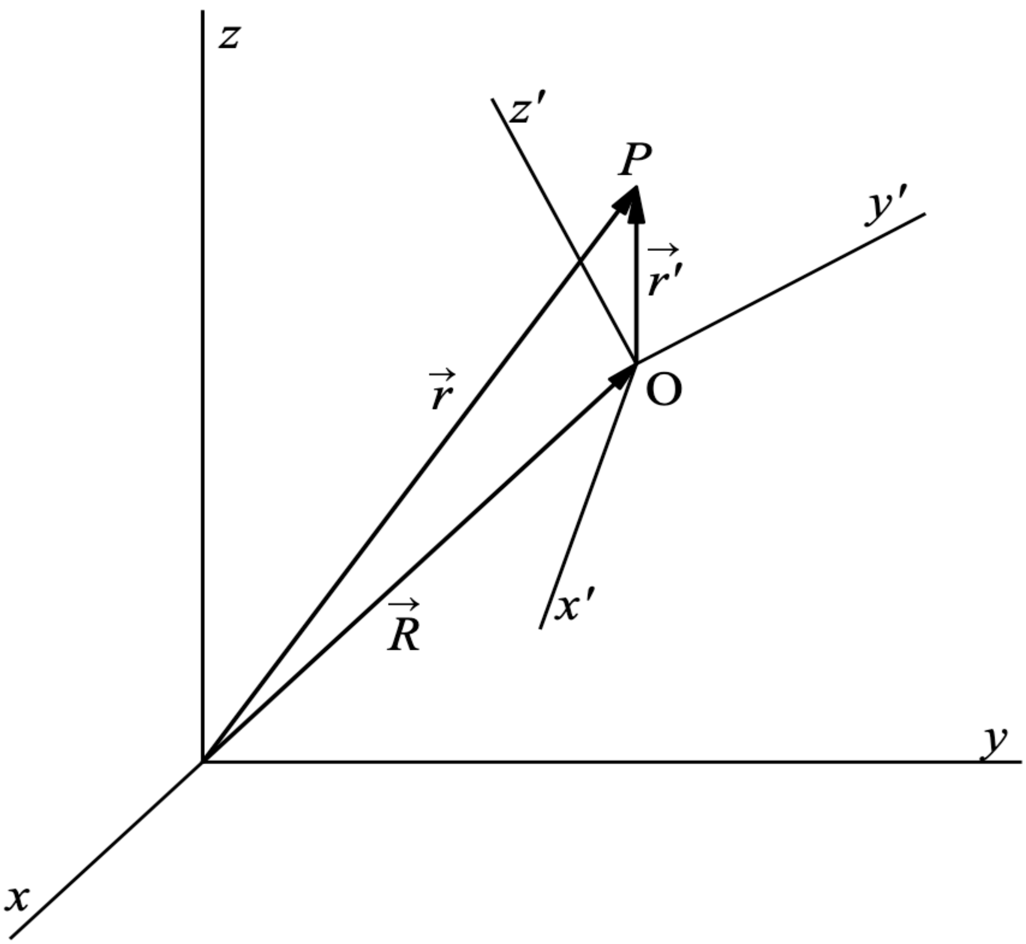
where $\vec{V}$ is the velocity of the origin of K', a frame of reference attached to the body's center of mass, $\vec{\Omega} = \dot{\Phi}(t)\,\hat{\Phi}$ is the body's angular velocity (its instantaneous counter-clockwise rotation speed around some axis in the direction of a unit vector $\hat{\Phi}$) and $\vec{r}\,'$ is the distance from the center of mass (origin of K') to the point P.
b) Derive algebraically the same result of **a)** , using the fact that vectors are defined up to parallel translation and so $\vec{r}$ and $\vec{r}\,'$ are related by a rotation matrix $\omega_{a,b}$ which, as all rotation matrices, is orthogonal.

Solution

```
restart;
with(Physics) :
with(Physics:-Vectors) :
```

In the solution of this problem it is convenient to keep the dependency on time t explicit.

a) The position of any point P of the body is represented in K by $\vec{r}(t)$, and in K' by $\vec{r}'(t)$. The origin of K' is located at $\vec{R}$, so



That is,

```
r_(t) = R_(t) + r'_(t)
```

$$\vec{r}(t) = \vec{R}(t) + \vec{r}'(t) \quad \textbf{(535)}$$

Differentiating and introducing the velocity $\vec{v}'(t)$ in K'

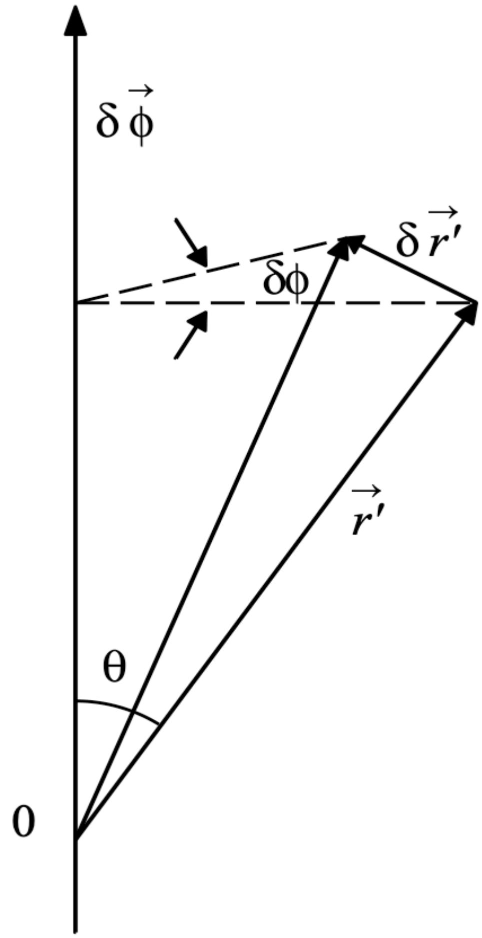
```
diff((535), t)
```

$$\dot{\vec{r}}(t) = \dot{\vec{R}}(t) + \dot{\vec{r}}'(t) \quad \textbf{(536)}$$

```
subs(diff(r'_(t), t) = v'_(t), (536))
```

$$\dot{\vec{r}}(t) = \dot{\vec{R}}(t) + \vec{v}'(t) \quad \textbf{(537)}$$

The vector $\vec{v}'(t)$ can be written as $\vec{\Omega}(t) \times \vec{r}'(t)$ as is clear from this drawing:



where $\delta\vec{r}' = \delta\vec{\phi} \times \vec{r}'$, so dividing by δt and denoting $\vec{\Omega} = \dfrac{\delta\vec{\phi}}{\delta t}$ we get

v'_(*t*) = *Omega*_(*t*) &x *r*'_(*t*)

$$\vec{v}'(t) = \vec{\Omega}(t) \times \vec{r}'(t) \tag{538}$$

Hence,

subs(**(538)**, **(537)**)

$$\dot{\vec{r}}(t) = \dot{\vec{R}}(t) + \vec{\Omega}(t) \times \vec{r}'(t) \tag{539}$$

---

b) This formula **(539)** can be derived algebraically in different ways. In what follows we opt for using tensor notation. First, the translation of the center of mass (vector $\vec{R}$) does not change the orientation of the axes of the frame K' rigidly attached to the body. To simplify things, then, assume the origins of K and K' coincide; the relation **(535)** between $\vec{r}$ and $\vec{r}'$ becomes

subs(*R*_(*t*) = 0, **(535)**)

$$\vec{r}(t) = \vec{r}'(t) \tag{540}$$

The body is assumed to be rotating around some axis (the direction of the vector $\delta \vec{\phi}$ of the drawing above), so the components of these two vectors $\vec{r}$ and $\vec{r'}$ are linearly related through a 3D rotation matrix. To express that relation using tensors, set *lowercaselatin* to denote *su3matrixindices*, and define four tensors with the components of the vectors $\vec{r}$, $\vec{r'}$, $\vec{\Omega}$ and the rotation matrix $\omega_{a,b}$

Setup(*su3matrixindices* = *lowercaselatin*, *tensors* = {*r*, *r*', Omega, omega})

$$\left[su3matrixindices = lowercaselatin, tensors = \left\{\Omega, \omega, r, r', \gamma_\mu, \sigma_\mu, \partial_\mu, g_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}\right\}\right] \quad (541)$$

Note that expanding $\vec{r}(t)$ in an orthogonal basis $\hat{r}_a$ attached to K

$$\vec{r}(t) = r_a(t) \cdot \hat{r}_a,$$

the components $r_a(t)$ depend on t while the unit vectors $\hat{r}_a$ of the basis are *constant*. On the other hand, expanding the same vector in K' the components are *constant* (the frame K' is rigidly attached to the body and P is a point of it) while the orientation of the basis of unit vectors $\hat{r'}_a$ of K', seen from K, varies with the time

$$\vec{r}(t) = r'_a \cdot \hat{r'}_a(t)$$

The relation $\vec{r}(t) = \vec{r'}(t)$ can then be written as a linear relation between the non-constant components $r_a(t)$ in K and the constant components r'_a in K' involving a 3D rotation matrix $\omega_{a,b}(t)$

r[*a*](*t*) = omega[*a*, *b*](*t*) *r*'[*b*]

$$r_a(t) = \omega_{a,b}(t)\, r'_b \quad (542)$$

Differentiating with respect to *t*,

diff(**(542)**, *t*)

$$\dot{r}_a(t) = \dot{\omega}_{a,b}(t)\, r'_b \quad (543)$$

The components r'_b on the right hand side can also be expressed in terms of $r_a(t)$ without the derivative $\dot{r}_a(t)$. For that, multiply **(542)** , *from the left*, by the inverse matrix $\omega_{b,a}^{-1}$. Since ω is a rotation matrix, it is orthogonal and so its inverse is equal to its transpose, $\omega_{b,a}^{-1} = \omega_{a,b}$,

omega[*a*, *b*](*t*)·**(542)**

$$\omega_{a,b}(t)\, r_a(t) = \omega_{a,b}(t)\, \omega_{a,c}(t)\, r'_c \quad (544)$$

Introducing the fact that $\omega_{a,b}$ and $\omega_{b,a}$ are inverses of each other,

omega[*b*, *a*](*t*)·omega[*b*, *c*](*t*) = *KroneckerDelta*[*a*, *c*]

$$\omega_{b,a}(t)\, \omega_{b,c}(t) = \delta_{a,c} \quad (545)$$

SubstituteTensor(**(545)**, **(544)**)

$$\omega_{a,b}(t)\, r_a(t) = \delta_{b,c}\, r'_c \quad (546)$$

Simplify(**(546)**)

$$\omega_{a,b}(t)\, r_a(t) = r'_b \quad (547)$$

resulting in the components r'_b expressed in terms of the $r_a(t)$

isolate(**(547)**, *r*'[*b*])

$$r'_b = \omega_{a,b}(t)\, r_a(t) \qquad \textbf{(548)}$$

For the velocity $\dot{r}_a(t)$ we then have

SubstituteTensor(**(548)**, **(543)**)

$$\dot{r}_a(t) = \dot{\omega}_{a,b}(t)\, \omega_{c,b}(t)\, r_c(t) \qquad \textbf{(549)}$$

Now, for every orthogonal matrix $\omega_{a,b}(t)$, the product $\dot{\omega}_{a,b}(t)\, \omega_{b,c}(t)$ is antisymmetric. To see that, consider again $\omega \cdot \omega^{-1} = \mathbb{I}$,

omega$[a, b](t)\cdot$omega$[c, b](t) = \mathit{KroneckerDelta}[a, c]$

$$\omega_{a,b}(t)\, \omega_{c,b}(t) = \delta_{a,c} \qquad \textbf{(550)}$$

Differentiating,

diff(**(550)**, t)

$$\dot{\omega}_{a,b}(t)\, \omega_{c,b}(t) + \omega_{a,b}(t)\, \dot{\omega}_{c,b}(t) = 0 \qquad \textbf{(551)}$$

The sum on the left-hand side is actually the *symmetric part of* $\dot{\omega}_{b,a}(t)\, \omega_{c,b}(t)$

op(1, *lhs*(**(551)**))

$$\dot{\omega}_{a,b}(t)\, \omega_{c,b}(t) \qquad \textbf{(552)}$$

Symmetrize(**(552)**) $\cdot$ 2

$$\dot{\omega}_{a,b}(t)\, \omega_{c,b}(t) + \omega_{a,b}(t)\, \dot{\omega}_{c,b}(t) \qquad \textbf{(553)}$$

and since the symmetric part of $\dot{\omega}_{b,a}(t)\, \omega_{c,b}(t)$ is equal to 0 (equation **(551)**) if follows that $\dot{\omega}_{a,b}(t)\, \omega_{c,b}(t)$ *is antisymmetric* in *a,c*. Then, $\dot{\omega}_{a,b}(t)\, \omega_{c,b}(t)$ can be expressed in terms of the totally antisymmetric LeviCivita tensor times another tensor of only one index, the components of the rotation vector $\vec{\Omega}$, defined as a tensor at the beginning in **(541)**

(552) $= -\mathit{LeviCivita}[a, c, b]\cdot$Omega$[b]$

$$\dot{\omega}_{a,b}(t)\, \omega_{c,b}(t) = \epsilon_{a,b,c}\, \Omega_b \qquad \textbf{(554)}$$

SubstituteTensor(**(554)**, **(549)**)

$$\dot{r}_a(t) = \epsilon_{a,b,c}\, \Omega_b\, r_c(t) \qquad \textbf{(555)}$$

Multiplying by the basis of unit vectors $\hat{r}_a$

$\_r\_[a]\cdot$**(555)**

$$\hat{r}_a\, \dot{r}_a(t) = \hat{r}_a\, \epsilon_{a,b,c}\, \Omega_b\, r_c(t) \qquad \textbf{(556)}$$

This result can be written using vector notation according to $\hat{r}_a \cdot \dot{r}_a(t) = \dot{\vec{r}}(t)$ and $\hat{r}_a \cdot \epsilon_{a,b,c} \cdot \Omega_b \cdot r_c(t) = \vec{\Omega}(t) \times \vec{r}(t)$

subs$\left(\dot{r}_a(t) = \dot{\vec{r}}(t), \epsilon_{a,b,c}\, \Omega_b\, r_c(t) = \vec{\Omega}(t) \times \vec{r}(t), \textbf{(555)}\right)$

$$\dot{\vec{r}}(t) = \vec{\Omega}(t) \times \vec{r}(t) \qquad \textbf{(557)}$$

and since in this calculation the origins of K and K' coincide, $\vec{r}(t) = \vec{r}'(t)$

subs ($r\_ = r'\_$, **(557)**)

$$\dot{\vec{r}}'(t) = \vec{\Omega}(t) \times \vec{r}'(t) \qquad \textbf{(558)}$$

Finally, substituting this result into **(536)** $\equiv \dot{\vec{r}}(t) = \dot{\vec{R}}(t) + \dot{\vec{r}}'(t)$,

subs (**(558)**, **(536)**)

$$\dot{\vec{r}}(t) = \dot{\vec{R}}(t) + \vec{\Omega}(t) \times \vec{r}'(t) \qquad \textbf{(559)}$$

which is the formula we wanted to derive.

Inertia tensor

Problem

a) Show that using $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}'$, derived in Angular velocity for the velocity $\vec{v}$ of a point P of a rigid body in terms of $\vec{\Omega}$ and the position $\vec{r}'$ of P viewed from the center of mass $\vec{R}$, the kinetic energy of the rigid body can be written in terms of the positions $\vec{r}'$ of the n particles (not their velocities), the velocity of the center of mass $\vec{V}$ and angular velocity $\vec{\Omega}$ as

$$T = \frac{1}{2} \left\|\vec{V}\right\|^2 \mu + \sum_{i=1}^{n} \left(\frac{1}{2} m_i \left\|\vec{\Omega}\right\|^2 \left\|\vec{r}'_i\right\|^2 - \frac{1}{2} m_i \left(\vec{\Omega} \cdot \vec{r}'_i\right)^2 \right)$$

b) Use tensor notation to show that this result can be rewritten as

$$T = \frac{1}{2} \left\|\vec{V}(t)\right\|^2 \mu + \frac{1}{2} \mathbb{I}_{a,b}\, \Omega_b\, \Omega_a$$

where

$$\mathbb{I}_{a,b} = \sum_{i=1}^{n} m_i \left({r_{i,c}}^2 \delta_{a,b} - r'_{i,a} r'_{i,b} \right)$$

is the Inertia tensor, Ω_a represents the components of the vector $\vec{\Omega}$ and $r'_{i,a}$ represents the components of the position vector $\vec{r}'_i$ of the i^{th} particle.

Solution

restart;

with (*Physics:-Vectors*) :

CompactDisplay (($r\_$, $r'\_$, *Omega_*, $R\_$, $V\_$) (t))

$\vec{r}(t)$ *will now be displayed as* $\vec{r}$

$\vec{r}'(t)$ *will now be displayed as* $\vec{r}'$

$\vec{\Omega}(t)$ *will now be displayed as* $\vec{\Omega}$

$\vec{R}(t)$ *will now be displayed as* $\vec{R}$

$\vec{V}(t)$ *will now be displayed as* $\vec{V}$ **(560)**

a) The kinetic energy in K is

$T = \frac{1}{2}\ Sum\left(m[i] \cdot diff(r\_[i](t), t)^2, i = 1..n\right)$

$$T = \frac{\left(\sum_{i=1}^{n} m_i \dot{\vec{r}}_i^{\,2}\right)}{2} \tag{561}$$

where the system is assumed to be discrete an consisting of n particles, each of which has mass m_i. From **(539)**, each pair $\left(\vec{r}_i, \vec{r'}_i\right)$ are related through

$subs\,(r\_ = r\_[i], r'\_(t) = r'\_[i](t), \textbf{(539)})$

$$\dot{\vec{r}}_i = \dot{\vec{R}} + \vec{\Omega} \times \vec{r'}_i \tag{562}$$

where as in the previous problem $\vec{R}$ is the position of the center of mass viewed from K. Substituting this relation into the expression for the kinetic energy,

$subs\,(\textbf{(562)}, \textbf{(561)})$

$$T = \frac{\left(\sum_{i=1}^{n} m_i \left(\dot{\vec{R}} + \vec{\Omega} \times \vec{r'}_i\right)^2\right)}{2} \tag{563}$$

Expanding the sum, power and vector products all in one go

$expand\,(\textbf{(563)})$

$$T = \frac{\|\dot{\vec{R}}\|^2 \left(\sum_{i=1}^{n} m_i\right)}{2} + \dot{\vec{R}} \cdot \left(\sum_{i=1}^{n} m_i \left(\vec{\Omega} \times \vec{r'}_i\right)\right) + \frac{\|\vec{\Omega}\|^2 \left(\sum_{i=1}^{n} m_i \|\vec{r'}_i\|^2\right)}{2} - \frac{\left(\sum_{i=1}^{n} m_i \left(\vec{\Omega} \cdot \vec{r'}_i\right)^2\right)}{2} \tag{564}$$

Denoting $\left(\sum_{i=1}^{n} m_i\right) = \mu$ and introducing the velocity of the center of mass (copy the sum from above, paste, then edit)

$subs\left(\sum_{i=1}^{n} m_i = \mu, diff\,(R\_(t), t) = V\_(t), \textbf{(564)}\right)$

$$T = \frac{\|\vec{V}\|^2 \mu}{2} + \vec{V} \cdot \left(\sum_{i=1}^{n} m_i \left(\vec{\Omega} \times \vec{r'}_i\right)\right) + \frac{\|\vec{\Omega}\|^2 \left(\sum_{i=1}^{n} m_i \|\vec{r'}_i\|^2\right)}{2} - \frac{\left(\sum_{i=1}^{n} m_i \left(\vec{\Omega} \cdot \vec{r'}_i\right)^2\right)}{2} \tag{565}$$

The second term of the right-hand side is equal to zero:

$op\,(2, rhs\,(\textbf{(565)}))$

$$\vec{V} \cdot \left(\sum_{i=1}^{n} m_i \left(\vec{\Omega} \times \vec{r'}_i\right)\right) \tag{566}$$

$Omega\_(t) \times r'\_[i](t) = -$ `%&x`$(r'\_[i](t), \%Omega\_(t))$

$$\vec{\Omega} \times \vec{r'}_i = -\vec{r'}_i \times \vec{\Omega} \tag{567}$$

were in the above we use the inert cross product `%&x` to keep the automatic reordering of the vectors being multiplied

subs((567), (566))

$$\vec{V} \cdot \left(\sum_{i=1}^{n} \left(-m_i \left(\vec{r'}_i \times \vec{\Omega} \right) \right) \right) \tag{568}$$

Expanding now the product,

expand((568))

$$\vec{V} \left(\vec{\Omega} \times \left(\sum_{i=1}^{n} m_i \vec{r'}_i \right) \right) \tag{569}$$

and since the origin of K' is at the center of mass, $\sum_{i=1}^{n} m_i \vec{r'}_i = 0$. The kinetic energy becomes

subs((566) = 0, (565))

$$T = \frac{\|\vec{V}\|^2 \mu}{2} + \frac{\|\vec{\Omega}\|^2 \left(\sum_{i=1}^{n} m_i \|\vec{r'}_i\|^2 \right)}{2} - \frac{\left(\sum_{i=1}^{n} m_i \left(\vec{\Omega} \cdot \vec{r'}_i \right)^2 \right)}{2} \tag{570}$$

Combining the sums, and at the same time factor out $\frac{1}{2}$ and m_i, instead of *combine*((570), *Sum*) one can use

$\frac{1}{2} \cdot (combine(2 \cdot \mathbf{(570)}, Sum))$

$$T = \frac{\left(\sum_{i=1}^{n} \left(m_i \|\vec{\Omega}\|^2 \|\vec{r'}_i\|^2 - m_i \left(\vec{\Omega} \cdot \vec{r'}_i \right)^2 \right) \right)}{2} + \frac{\|\vec{V}\|^2 \mu}{2} \tag{571}$$

simplify((571))

$$T = \frac{\left(\sum_{i=1}^{n} \left(m_i \|\vec{\Omega}\|^2 \|\vec{r'}_i\|^2 - m_i \left(\vec{\Omega} \cdot \vec{r'}_i \right)^2 \right) \right)}{2} + \frac{\|\vec{V}\|^2 \mu}{2} \tag{572}$$

This is the expected result, factored out as traditionally shown in textbooks.

---

b) In order to rewrite this result **(571)** for the kinetic energy T in tensorial form, load *Physics* for the tools for tensor computations

with(*Physics*) :

The problem at hands is one of representation. The vector $\vec{\Omega}$ can be represented by a tensor Ω_a with a *spaceindex* a running from 1 to the 3 corresponding to each of the components of $\vec{\Omega}$. Representing the position vector $\vec{r'}_i$ using tensor notation, however, is trickier: there is one index, a, that also runs through the space dimensions corresponding to the components of $\vec{r'}_i$ but there is also another index, i, identifying a particle among n of them, where n is abstract. To represent the index i, one can use tensors with *genericindices*, which run from 1 to a generic unspecified dimension. So the components of $\vec{r'}_i$ can be properly represented by a tensor with two indices of different kinds: $r'_{i,\, a}$.

So set lowercase latin indices from a to h to represent *spaceindices* and lowercase latin from i to s representing *genericindices*. On the way, set the tensors representing the components of $\vec{\Omega}$ and $\vec{r'}_i$

$Setup(spaceindices = lowercase\_ah, generic = lowercase\_is, tensors = \{\text{Omega}[a], r'[i, a]\})$

* *Partial match of 'generic' against keyword 'genericindices'*

$$\left[genericindices = lowercaselatin\_is, spaceindices = lowercaselatin\_ah, tensors = \left\{\gamma_{\mu}, \Omega_{a}, \sigma_{\mu}, \partial_{\mu}, g_{\mu,\, \nu}, \gamma_{a,\, b}, r'_{i,\, a}, \epsilon_{\alpha,\, \beta,\, \mu,\, \nu}\right\}\right] \quad \textbf{(573)}$$

Recalling the result of item **a)**

(572)

$$T = \frac{\left(\sum_{i=1}^{n} \left(m_i \left\|\vec{\Omega}\right\|^2 \left\|\vec{r'}_i\right\|^2 - m_i \left(\vec{\Omega} \cdot \vec{r'}_i\right)^2\right)\right)}{2} + \frac{\left\|\vec{V}\right\|^2 \mu}{2} \quad \textbf{(574)}$$

The product

$Omega\_(t) \cdot r'\_i(t)$

$$\vec{\Omega} \cdot \vec{r'}_i \quad \textbf{(575)}$$

can be represented as

(575) $= \text{Omega}[a] \cdot r'[i, a]$

$$\vec{\Omega} \cdot \vec{r'}_i = \Omega_a\, r'_{i,\, a} \quad \textbf{(576)}$$

where for simplicity we omitted the time dependency of $\Omega_a(t)$ since in this problem there is no need for differentiating with respect to time, and recall that the components $r'_{i,\, a}$ of the position vector o a point of the body - measured from the K' which is a frame rigidly attached to the body - are constant. Squaring,

(576)^2

$$\left(\vec{\Omega} \cdot \vec{r'}_i\right)^2 = \Omega_a\, r'_{i,\, a}\, \Omega_b\, r'_{i,\, b} \quad \textbf{(577)}$$

For $\left\|\vec{r'}_i\right\|^2$ and $\left\|\vec{\Omega}\right\|^2$

$Norm\left(r'\_i(t)\right)^2 = r'[i, c]^2$

$$\left\|\vec{r'}_i\right\|^2 = {r'}_{i,\, c}^{2} \quad \textbf{(578)}$$

$Norm(Omega\_(t))^2 = \text{Omega}[a]^2$

$$\left\|\vec{\Omega}\right\|^2 = \Omega_a^2 \quad \textbf{(579)}$$

Having in mind that the result we want includes factoring out $\Omega_a \cdot \Omega_b$, rewrite the right-hand side of the above as

rhs((**579**)) = *KroneckerDelta*[*a*, *b*] Omega[*a*] Omega[*b*]

$$\Omega_a^2 = \delta_{a,b}\,\Omega_a\,\Omega_b \tag{580}$$

Substituting *sequentially* all these vector → tensor relations,

subs((**577**), (**578**), (**579**), (**580**), (**571**))

$$T = \frac{\left(\sum_{i=1}^{n} \left(m_i\,\delta_{a,b}\,\Omega_a\,\Omega_b\,{r'}_{i,c}^{2} - m_i\,\Omega_a\,{r'}_{i,a}\,\Omega_b\,{r'}_{i,b}\right)\right)}{2} + \frac{\|\vec{V}\|^2\,\mu}{2} \tag{581}$$

Expanding, collecting Ω and combining the coefficients we get

expand((**581**))

$$T = \frac{\delta_{a,b}\,\Omega_a\,\Omega_b\left(\sum_{i=1}^{n} m_i\,{r'}_{i,c}^{2}\right)}{2} - \frac{\Omega_a\,\Omega_b\left(\sum_{i=1}^{n} m_i\,{r'}_{i,a}\,{r'}_{i,b}\right)}{2} + \frac{\|\vec{V}\|^2\,\mu}{2} \tag{582}$$

collect((**582**), [Omega[*a*], Omega[*b*]], *combine*)

$$T = \left(\sum_{i=1}^{n}\left(\frac{1}{2}\,m_i\,{r'}_{i,c}^{2}\,\delta_{a,b} - \frac{1}{2}\,m_i\,{r'}_{i,a}\,{r'}_{i,b}\right)\right)\Omega_b\,\Omega_a + \frac{\|\vec{V}\|^2\,\mu}{2} \tag{583}$$

To get this result in more compact form as frequently shown in textbooks one can tweak the input above as

$\frac{1}{2}$ · *collect*(2·(**582**), [Omega[*a*], Omega[*b*]], *factor*@*combine*)

$$T = \frac{\left(\sum_{i=1}^{n}\left(-m_i\left(-{r'}_{i,c}^{2}\,\delta_{a,b} + {r'}_{i,a}\,{r'}_{i,b}\right)\right)\right)\Omega_b\,\Omega_a}{2} + \frac{\|\vec{V}\|^2\,\mu}{2} \tag{584}$$

where the coefficient of the product $\Omega_a\,\Omega_b$ times 2 is the inertia tensor:

$\mathbb{I}_{a,b}$ = 2· *Coefficients*(*rhs*((**584**)), Omega[*a*] Omega[*b*], 1)

$$\mathbb{I}_{a,b} = \sum_{i=1}^{n}\left(-m_i\left(-{r'}_{j,c}^{2}\,\delta_{a,b} + {r'}_{i,a}\,{r'}_{i,b}\right)\right) \tag{585}$$

subs($j = i$, (**585**))

$$\mathbb{I}_{a,b} = \sum_{i=1}^{n}\left(-m_i\left(-{r'}_{i,c}^{2}\,\delta_{a,b} + {r'}_{i,a}\,{r'}_{i,b}\right)\right) \tag{586}$$

and T can be written as

Substitute((*rhs* = *lhs*)((**586**)), (**584**))

$$T = \frac{\mathbb{I}_{a,b}\,\Omega_b\,\Omega_a}{2} + \frac{\|\vec{V}\|^2\,\mu}{2} \tag{587}$$

Problem

Determine the Inertia tensor corresponding to a triatomic molecule that has the form of an isosceles triangle with two masses m_1 in the extremes of the base and a mass m_2 in the third vertex. The distance between the two masses m_1 is equal to *a,* and the height of the triangle is equal to *h*.

Solution

restart;
with(*Physics*, *KroneckerDelta*) :
with(*Physics*:-*Vectors*) :

The general formula for the Inertia tensor is given by (note the use of the abbreviation *kd_* for *KroneckerDelta*)

InertiaTensor := *%sum*(*m*[*k*] (*Norm*(*r_*[*k*])^2 *kd_*[*i*, *j*] − *Component*(*r_*[*k*], *i*) *Component*(*r_*[*k*], *j*)), *k* = 1 ..*N*)

$$InertiaTensor := \sum_{k=1}^{N} m_k \left(\left\|\vec{r}_k\right\|^2 \delta_{i,j} - \left(\vec{r}_k\right)_i \left(\vec{r}_k\right)_j \right) \quad \textbf{(588)}$$

where N is the number of particles, m_k is the mass of each particle, $\vec{r}_k$ is its position in a reference system with the origin at the "center of mass", $\left(\vec{r}_k\right)_i$ is the component in the i^{th} direction of the position vector associated to the k^{th} particle in the reference system, and $\delta_{i,j}$ is the Kronecker delta, part of Physics. Set this definition of the *InertiaTensor* to be an indexing function for the *InertiaTensor* matrix.

IT := *unapply*(*InertiaTensor*, *i*, *j*)

$$IT := (i, j) \mapsto \sum_{k=1}^{N} m_k \cdot \left(\left\|\vec{r}_k\right\|^2 \cdot \delta_{i,j} - \left(\vec{r}_k\right)_i \cdot \left(\vec{r}_k\right)_j \right) \quad \textbf{(589)}$$

For example, for a component in the diagonal, we have

IT(1, 1)

$$\sum_{k=1}^{N} m_k \left(\left\|\vec{r}_k\right\|^2 - \left(\vec{r}_k\right)_1^{\,2} \right) \quad \textbf{(590)}$$

and outside of the diagonal we have

IT(1, 2)

$$\sum_{k=1}^{N} \left(-m_k \left(\vec{r}_k\right)_1 \left(\vec{r}_k\right)_2 \right) \quad \textbf{(591)}$$

At this point we can proceed to setting the particularities of this problem. The number of particles is 3.

N := 3

$$N := 3 \quad \textbf{(592)}$$

Hence the matrix is

IT_Matrix := *Matrix*(3, 3, *IT*)

IT_Matrix := **(593)**

$$\begin{bmatrix} \sum_{k=1}^{3} m_k \left(\|\vec{r}_k\|^2 - (\vec{r}_k)_1^{\;2} \right) & \sum_{k=1}^{3} \left(-m_k (\vec{r}_k)_1 (\vec{r}_k)_2 \right) & \sum_{k=1}^{3} \left(-m_k (\vec{r}_k)_1 (\vec{r}_k)_3 \right) \\ \sum_{k=1}^{3} \left(-m_k (\vec{r}_k)_1 (\vec{r}_k)_2 \right) & \sum_{k=1}^{3} m_k \left(\|\vec{r}_k\|^2 - (\vec{r}_k)_2^{\;2} \right) & \sum_{k=1}^{3} \left(-m_k (\vec{r}_k)_2 (\vec{r}_k)_3 \right) \\ \sum_{k=1}^{3} \left(-m_k (\vec{r}_k)_1 (\vec{r}_k)_3 \right) & \sum_{k=1}^{3} \left(-m_k (\vec{r}_k)_2 (\vec{r}_k)_3 \right) & \sum_{k=1}^{3} m_k \left(\|\vec{r}_k\|^2 - (\vec{r}_k)_3^{\;2} \right) \end{bmatrix}$$

Two of the masses are equal

$m[3] := m[1]$

$$m_3 := m_1 \tag{594}$$

Now choose any system of reference (not at the center of mass) where we are going to project the position vectors $\vec{R}_k$ of each atom as well as the center of mass $\vec{R}_{CM}$. The vectors $\vec{r}_k$ entering the definition of the Inertia tensor **(588)** are related to $\vec{R}_k$ and $\vec{R}_{CM}$ by

$position := r\_[k] = R\_[k] - R\_[CM]$

$$position := \vec{r}_k = \vec{R}_k - \vec{R}_{CM} \tag{595}$$

For $\vec{R}_k$, we choose a system of reference with the origin at the middle of the segment connecting the two atoms of mass equal to m_1. Using Cartesian coordinates, we take the x axis along this segment and the z axis passing through the third atom of mass m_2. So in this referential, the positions of the three atoms are

$R\_[1] := -\frac{a}{2}\ \_i$ *# to the left of the origin*

$$\vec{R}_1 := -\frac{a\,\hat{i}}{2} \tag{596}$$

$R\_[2] := h\ \_k$ *# along the z direction*

$$\vec{R}_2 := h\,\hat{k} \tag{597}$$

$R\_[3] := \frac{a}{2}\ \_i$ *# to the right of the origin*

$$\vec{R}_3 := \frac{a\,\hat{i}}{2} \tag{598}$$

Indicate the real objects of this problem so that simplification steps further below can take that into account

$Setup(real = \{a, h, m[1], m[2], m[3]\})$

** Partial match of 'real' against keyword 'realobjects'*

$$[realobjects = \{a, h, \phi, r, \rho, \theta, x, y, z, m_1, m_2\}] \tag{599}$$

Compute the position of the "*center of mass*." By definition, it is

$R\_[CM] := \frac{\%sum(m[k]\,R\_[k],\ k = 1..N)}{\%sum(m[k],\ k = 1..N)}$

$$\vec{R}_{CM} := \frac{\sum_{k=1}^{3} m_k \vec{R}_k}{\sum_{k=1}^{3} m_k} \quad \textbf{(600)}$$

Evaluating these sums, we have the value of $\vec{R}_{CM}$:

$R\_[CM] := value(R\_[CM])$

$$\vec{R}_{CM} := \frac{m_2 h \hat{k}}{2 m_1 + m_2} \quad \textbf{(601)}$$

So these are the positions of the three particles viewed from the center of mass and expressed in terms of the known quantities m_k, a and h:

$seq(eval(position, k=j), j=1..N)$

$$\vec{r}_1 = -\frac{a \hat{i}}{2} - \frac{m_2 h \hat{k}}{2 m_1 + m_2}, \vec{r}_2 = h \hat{k} - \frac{m_2 h \hat{k}}{2 m_1 + m_2}, \vec{r}_3 = \frac{a \hat{i}}{2} - \frac{m_2 h \hat{k}}{2 m_1 + m_2} \quad \textbf{(602)}$$

The answer to the problem posed, that is the inertia tensor for this triatomic molecule, is now obtained by evaluating the abstract expression for the *IT_Matrix* at these values of the position vectors $\vec{r}_k$

$IT\_answer := simplify(eval(value(IT\_Matrix), [\textbf{(602)}]))$

$$IT\_answer := \begin{bmatrix} \frac{2 m_2 h^2 m_1}{2 m_1 + m_2} & 0 & 0 \\ 0 & \frac{2 a^2 m_1^2 + m_2 (a^2 + 4 h^2) m_1}{4 m_1 + 2 m_2} & 0 \\ 0 & 0 & \frac{m_1 a^2}{2} \end{bmatrix} \quad \textbf{(603)}$$

Angular momentum of a rigid body

In the section related to the conservation of angular momentum, the solution to the second **Problem** shows that the value of the angular momentum $\vec{L}$ of a system of particles depends on the origin of the frame of reference. In this section, it is assumed that the origin is at the center of mass, so $\sum_{i=1}^{n} m_i \vec{r}_i = 0$.

Problem

Show, using tensor notation, that in the K' system whose origin is at the center of mass, the components L'_a of the angular momentum of a rigid body can be expressed in terms of the inertia tensor $\mathbb{I}_{a,b}$ and the components of the angular velocity Ω_b as

$$L'_a = I_{a,b}\,\Omega_b$$

Solution

restart;
with(*Physics*) : *with*(*Vectors*) :
CompactDisplay((*r_*, *r'_*, *Omega_*, *R_*, *V_*)(*t*))

$$\vec{r}(t) \textit{ will now be displayed as } \vec{r}$$
$$\vec{r'}(t) \textit{ will now be displayed as } \vec{r'}$$
$$\vec{\Omega}(t) \textit{ will now be displayed as } \vec{\Omega}$$
$$\vec{R}(t) \textit{ will now be displayed as } \vec{R}$$
$$\vec{V}(t) \textit{ will now be displayed as } \vec{V} \quad \textbf{(604)}$$

Denoting $\vec{r'}_i$ the position vectors of each point of the body viewed from K', the expression of the angular momentum $\vec{L'}$ is given by

*L'_* = *Sum*(*m*[*i*]·*r'_*[*i*](*t*) × *diff*(*r'_*(*t*), *t*), *i* = 1 ..*n*)

$$\vec{L'} = \sum_{i=1}^{n} m_i \left(\vec{r'}_i \times \dot{\vec{r'}}\right) \quad \textbf{(605)}$$

In turn, in Angular velocity is is shown that $\vec{v'} = \vec{\Omega} \times \vec{r'}$

subs(*diff*(*r'_*(*t*), *t*) = *Omega_*(*t*) × *r'_*[*i*](*t*), **(605)**)

$$\vec{L'} = \sum_{i=1}^{n} m_i \left(\vec{r'}_i \times \left(\vec{\Omega} \times \vec{r'}_i\right)\right) \quad \textbf{(606)}$$

Expanding only the vector product

subsindets(**(606)**, *specfunc*(\`&x\`), *expand*)

$$\vec{L'} = \sum_{i=1}^{n} m_i \left(\vec{\Omega}\,\left\|\vec{r'}_i\right\|^2 - \left(\vec{\Omega}\cdot\vec{r'}_i\right)\vec{r'}_i\right) \quad \textbf{(607)}$$

Introducing tensor notation, as in item **b)** of this problem for the Inertia tensor,

Setup(*spaceindices* = *lowercase_ah*, *generic* = *lowercase_is*, *tensors* = {Omega[*a*], *r'*[*i*, *a*]})

** Partial match of 'generic' against keyword 'genericindices'*

$$\left[genericindices = lowercaselatin\_is,\ spaceindices = lowercaselatin\_ah,\ tensors = \left\{\gamma_\mu, \Omega_a, \sigma_\mu, \partial_\mu, g_{\mu,\nu}, \gamma_{a,b}, r'_{i,a}, \epsilon_{\alpha,\beta,\mu,\nu}\right\}\right] \quad \textbf{(608)}$$

*L'_* = *L'*[*a*], *Omega_*(*t*) . *r'_*[*i*](*t*) = Omega[*b*]·*r'*[*i*, *b*], *Norm*(*r'_*[*i*](*t*))^2 = *r'*[*i*, *c*]^2, *r'_*[*i*](*t*) = *r'*[*i*, *a*], *Omega_*(*t*) = *KroneckerDelta*[*a*, *b*]·Omega[*b*]

$$\vec{L'} = L'_a,\ \vec{\Omega}\cdot\vec{r'}_i = \Omega_b\, r'_{i,b},\ \left\|\vec{r'}_i\right\|^2 = {r'}_{i,c}^{\,2},\ \vec{r'}_i = r'_{i,a},\ \vec{\Omega} = \delta_{a,b}\,\Omega_b \quad \textbf{(609)}$$

subs(**(609)**, **(607)**)

$$L'_a = \sum_{i=1}^{n} m_i \left(\delta_{a,b}\, \Omega_b\, {r'}_{i,c}^{2} - \Omega_b\, r'_{i,b}\, r'_{i,a}\right) \tag{610}$$

Expanding, then collecting Ω and combining the sums in the coefficient
expand(**(610)**)

$$L'_a = \delta_{a,b}\, \Omega_b \left(\sum_{i=1}^{n} m_i\, {r'}_{i,c}^{2}\right) - \Omega_b \left(\sum_{i=1}^{n} m_i\, r'_{i,a}\, r'_{i,b}\right) \tag{611}$$

collect(**(611)**, Omega[*b*], *factor*@*combine*)

$$L'_a = \left(\sum_{i=1}^{n} \left(-m_i \left(-{r'}_{i,c}^{2}\, \delta_{a,b} + r'_{i,a}\, r'_{i,b}\right)\right)\right) \Omega_b \tag{612}$$

Introducing the expression for the inertia tensor,
(586)

$$\mathbb{I}_{a,b} = \sum_{i=1}^{n} \left(-m_i \left(-{r'}_{i,c}^{2}\, \delta_{a,b} + r'_{i,a}\, r'_{i,b}\right)\right) \tag{613}$$

subs((*rhs* = *lhs*)(**(586)**), **(612)**)

$$L'_a = \mathbb{I}_{a,b}\, \Omega_b \tag{614}$$

The equations of motion of a rigid body

A rigid body is a system with six degrees of freedom: three indicating the position $\vec{R}(t)$ plus three angles specifying the orientation of the axes of K' with respect to those of K. As discussed in the equations of motion for many-particle systems, the two vectorial equations of motion are
value(**(182)**)

$$\dot{\vec{P}}(t) = \vec{F}_{ext} \tag{615}$$

diff(*L_*(*t*), *t*) = *N_*(*t*)

$$\dot{\vec{L}}(t) = \vec{N}(t) \tag{616}$$

(190)

$$\vec{N} = \sum_{i=1}^{n} \vec{r}_i \times \vec{f}_{i,ext} \tag{617}$$

where $\vec{P}$ is the total momentum, $\vec{F}_{ext}$ is the total external force acting upon the body, $\vec{f}_{i,ext}$ is the external force acting upon the i^{th} particle, $\vec{L}$ the total angular momentum and $\vec{N}$ is the total torque.

Problem
Show that the equations of movement of a rigid body can be computed as the Lagrange equations for $\vec{R}$ and $\vec{\Phi}$ from the Lagrangian

$$L = \frac{1}{2}\,\mu\,\|\vec{V}\|^2 + \frac{1}{2}\,\mathbb{I}_{a,b}\,\Omega_b\,\Omega_a + U(\vec{R}, \vec{\Phi})$$

where $\mu = \sum_{i=1}^{n} m_i$ is the total mass, $\mathbb{I}_{a,b}$ is the inertia tensor, $U(\vec{R}, \vec{\Phi})$ is the potential energy for the external force $\vec{F}_{ext}$, and $\vec{\Omega} = \dot{\vec{\Phi}}$.

Solution

```
restart;
with(Physics:-Vectors) :
CompactDisplay((R_, V_, Phi_, Omega)(t))
```

$\vec{R}(t)$ *will now be displayed as* $\vec{R}$

$\vec{V}(t)$ *will now be displayed as* $\vec{V}$

$\vec{\Phi}(t)$ *will now be displayed as* $\vec{\Phi}$

$\Omega(t)$ *will now be displayed as* Ω **(618)**

The required derivation is easy by expressing the Lagrangian in tensor notation from the beginning. In what follows, however, with the purpose of illustrating different techniques, Lagrange's equations are computed using vectorial notation, only switching to tensor notation at the time of expressing the time derivative of the angular momentum.

The kinetic energy T in vectorial form is derived in this problem for the inertia tensor

```
subs(r'_[i](t) = r'_[i], (572))
```

$$T = \frac{\sum_{i=1}^{n}\left(m_i\,\|\vec{\Omega}(t)\|^2\,\|\vec{r'}_i\|^2 - m_i\,\left(\vec{\Omega}(t)\cdot\vec{r'}_i\right)^2\right)}{2} + \frac{\|\vec{V}\|^2\,\mu}{2} \qquad \textbf{(619)}$$

Adding the potential energy as a function of the *coordinates* $\vec{R}$ (location of the center of mass) and $\vec{\Phi}$ (the rotation axis, so that $\vec{\Omega} = \dot{\vec{\Phi}}$), the Lagrangian in vectorial form is given by

```
L = rhs((619)) + U(R_(t), Phi_(t))
```

$$L = \frac{\sum_{i=1}^{n}\left(m_i\,\|\vec{\Omega}(t)\|^2\,\|\vec{r'}_i\|^2 - m_i\,\left(\vec{\Omega}(t)\cdot\vec{r'}_i\right)^2\right)}{2} + \frac{\|\vec{V}\|^2\,\mu}{2} + U(\vec{R}, \vec{\Phi}) \qquad \textbf{(620)}$$

The first equation of movement, for the total momentum $\vec{P}(t)$ is derived as the Lagrange equation for this Lagrangian

```
%diff(%diff(L, V_(t)), t) = %diff(L, R_(t))
```

$$\frac{\partial}{\partial t}\,\frac{dL}{d\vec{V}} = \frac{dL}{d\vec{R}} \qquad \textbf{(621)}$$

```
subs((620), (621))
```

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \vec{V}}\left(\frac{\left(\sum_{i=1}^{n}\left(m_i \|\vec{\Omega}(t)\|^2 \|\vec{r'}_i\|^2 - m_i \left(\vec{\Omega}(t)\cdot\vec{r'}_i\right)^2\right)\right)}{2} + \frac{\|\vec{V}\|^2 \mu}{2} + U(\vec{R},\vec{\Phi})\right) = \frac{\partial}{\partial \vec{R}}\left(\frac{\left(\sum_{i=1}^{n}\left(m_i \|\vec{\Omega}(t)\|^2 \|\vec{r'}_i\|^2 - m_i \left(\vec{\Omega}(t)\cdot\vec{r'}_i\right)^2\right)\right)}{2} + \frac{\|\vec{V}\|^2 \mu}{2} + U(\vec{R},\vec{\Phi})\right) \quad \textbf{(622)}$$

value(%)

$$\mu \dot{\vec{V}} = \mathrm{D}_1(U)(\vec{R},\vec{\Phi}) \quad \textbf{(623)}$$

On the right-hand side, $\mathrm{D}_1(U)(\vec{R},\vec{\Phi})$ *is the derivative of* $U(\vec{R},\vec{\Phi})$ with respect to its first argument $\vec{R}$, equivalent to the gradient taking $\vec{R}$ as the coordinates, and $\mu\dot{\vec{V}} = \dot{\vec{P}}(t)$ is the total momentum

$subs\left(rhs(\textbf{(623)}) = F\_[ext](t), diff(V\_(t), t) = \frac{diff(P\_(t), t)}{\mathrm{mu}}, \textbf{(623)}\right)$

$$\dot{\vec{P}}(t) = \vec{F}_{ext}(t) \quad \textbf{(624)}$$

Note however that in Maple the Vectors:-Gradient command computes the gradient with respect to Cartesian, cylindrical or spherical coordinates, not an arbitrary vector $\vec{R}$. If more precision is required, the dependency of the potential energy could be expressed in terms of the norm of $\vec{R}$, as in

$U(Norm(R\_))$

$$U(\|\vec{R}\|) \quad \textbf{(625)}$$

in which case differentiating with respect to $\vec{R}$ is equivalent to the definition of directional derivative

$(\%diff = diff)\left(\textbf{(625)}, \vec{R}\right)$

$$\frac{\mathrm{d}}{\mathrm{d}\vec{R}} U(\|\vec{R}\|) = \frac{\mathrm{D}(U)(\|\vec{R}\|)\,\vec{R}}{\|\vec{R}\|} \quad \textbf{(626)}$$

The same computation, this time with respect to the *coordinates* $\vec{\Phi}$ where $\vec{\Omega} = \dot{\vec{\Phi}}$ is the corresponding velocity,

$\%diff(\%diff(L, Omega\_(t)), t) = \%diff(L, Phi\_(t))$

$$\frac{\partial}{\partial t}\frac{\mathrm{d}L}{\mathrm{d}\vec{\Omega}(t)} = \frac{\mathrm{d}L}{\mathrm{d}\vec{\Phi}} \quad \textbf{(627)}$$

$subs(\textbf{(620)}, \textbf{(627)})$

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \vec{\Omega}(t)}\left(\frac{\left(\sum_{i=1}^{n}\left(m_i \|\vec{\Omega}(t)\|^2 \|\vec{r'}_i\|^2 - m_i \left(\vec{\Omega}(t)\cdot\vec{r'}_i\right)^2\right)\right)}{2} + \frac{\|\vec{V}\|^2 \mu}{2} + U(\vec{R},\vec{\Phi})\right) = \frac{\partial}{\partial \vec{\Phi}}\left(\frac{\left(\sum_{i=1}^{n}\left(m_i \|\vec{\Omega}(t)\|^2 \|\vec{r'}_i\|^2 - m_i \left(\vec{\Omega}(t)\cdot\vec{r'}_i\right)^2\right)\right)}{2} + \frac{\|\vec{V}\|^2 \mu}{2} + U(\vec{R},\vec{\Phi})\right) \quad \textbf{(628)}$$

$subs(sum = Sum, value(\textbf{(628)}))$

$$\frac{\left(\sum_{i=1}^{n}\left(2\,m_i\,\left\|\vec{r'}_i\right\|^2\,\dot{\vec{\Omega}}(t)-2\,m_i\,\left(\dot{\vec{\Omega}}(t)\cdot\vec{r'}_i\right)\vec{r'}_i\right)\right)}{2}=\mathrm{D}_2(U)\left(\vec{R},\vec{\Phi}\right) \tag{629}$$

Switching to tensor notation as done in the other problems, the left-hand side can be rewritten as the time derivative of the components L'_a of the angular momentum .

```
with(Physics) :
Setup(spaceindices = lowercase_ah, generic = lowercase_is, tensors = {Omega[a], r'[i, a]})
```

** Partial match of 'generic' against keyword 'genericindices'*

$$\left[genericindices=lowercaselatin\_is,\ spaceindices=lowercaselatin\_ah,\ tensors=\left\{\gamma_\mu,\Omega_a,\sigma_\mu,\partial_\mu,\ g_{\mu,\nu},\gamma_{a,b},r'_{i,a},\epsilon_{\alpha,\beta,\mu,\nu}\right\}\right] \tag{630}$$

Introducing tensor components for the vectors of **(629)**

```
diff(Omega_(t), t) . r'_[i] = diff(Omega[b](t), t)·r'[i, b],
diff(Omega_(t), t) = KroneckerDelta[a, b]·diff(Omega[b](t) , t),
Norm(r'_[i])^2 = r'[i, c]^2,
r'_[i] = r'[i, a],
D[2](U)(R_(t), Phi_(t)) = Component(D[2](U)(R_(t), Phi_(t)), a)
```

$$\dot{\vec{\Omega}}(t)\cdot\vec{r'}_i=\dot{\Omega}_b\,r'_{i,b},\ \dot{\vec{\Omega}}(t)=\delta_{a,b}\,\dot{\Omega}_b,\ \left\|\vec{r'}_i\right\|^2=r'^2_{i,c},\ \vec{r'}_i=r'_{i,a},\ \mathrm{D}_2(U)\left(\vec{R},\vec{\Phi}\right)=\left(\mathrm{D}_2(U)\left(\vec{R},\vec{\Phi}\right)\right)_a \tag{631}$$

eval(**(629)**, [**(631)**])

$$\frac{\left(\sum_{i=1}^{n}\left(2\,m_i\,r'^2_{i,c}\,\delta_{a,b}\,\dot{\Omega}_b-2\,m_i\,\dot{\Omega}_b\,r'_{i,b}\,r'_{i,a}\right)\right)}{2}=\left(\mathrm{D}_2(U)\left(\vec{R},\vec{\Phi}\right)\right)_a \tag{632}$$

expand(**(632)**)

$$\delta_{a,b}\,\dot{\Omega}_b\left(\sum_{i=1}^{n}m_i\,r'^2_{i,c}\right)-\dot{\Omega}_b\left(\sum_{i=1}^{n}m_i\,r'_{i,a}\,r'_{i,b}\right)=\left(\mathrm{D}_2(U)\left(\vec{R},\vec{\Phi}\right)\right)_a \tag{633}$$

collect(**(633)**, diff(Omega[b](t), t), factor@combine)

$$\left(\sum_{i=1}^{n}\left(-m_i\left(-r'^2_{i,c}\,\delta_{a,b}+r'_{i,a}\,r'_{i,b}\right)\right)\right)\dot{\Omega}_b=\left(\mathrm{D}_2(U)\left(\vec{R},\vec{\Phi}\right)\right)_a \tag{634}$$

Introducing the expression for the inertia tensor,
(586)

$$\mathcal{I}_{a,b}=\sum_{i=1}^{n}\left(-m_i\left(-r'^2_{i,c}\,\delta_{a,b}+r'_{i,a}\,r'_{i,b}\right)\right) \tag{635}$$

subs((rhs = lhs)(**(586)**), **(634)**)

$$\mathcal{I}_{a,b}\,\dot{\Omega}_b=\left(\mathrm{D}_2(U)\left(\vec{R},\vec{\Phi}\right)\right)_a \tag{636}$$

From the result **(614)** for the problem of representing the angular momentum of a rigid body in terms of the inertia tensor the left-hand side is the derivative of the components of the angular momentum
(614)

$$L'_a = \mathbb{I}_{a,b}\,\Omega_b \quad \textbf{(637)}$$

subs (Omega[b] = Omega[b](t), L'[a] = L'[a](t), **(614)**)

$$L'_a(t) = \mathbb{I}_{a,b}\,\Omega_b \quad \textbf{(638)}$$

diff (**(638)**, t)

$$\dot{L}'_a(t) = \mathbb{I}_{a,b}\,\dot{\Omega}_b \quad \textbf{(639)}$$

subs ((*rhs* = *lhs*) (**(639)**), **(636)**)

$$\dot{L}'_a(t) = \left(\mathrm{D}_2(U)\left(\vec{R}, \vec{\Phi}\right)\right)_a \quad \textbf{(640)}$$

The right-hand side is the a^{th} component of the variation of the potential energy $U\left(\vec{R}, \vec{\Phi}\right)$ with respect to $\vec{\Phi}$. A graphics analysis of $\mathrm{D}_2(U)\left(\vec{R}, \vec{\Phi}\right)$ and algebraic derivation as done in the problem of the section Angular velocity results in

D$_2$(U)(R_(t), *Phi*_(t)) = R_(t) × F_[*ext*]

$$\mathrm{D}_2(U)\left(\vec{R}, \vec{\Phi}\right) = \vec{R} \times \vec{F}_{ext} \quad \textbf{(641)}$$

subs (**(641)**, **(640)**)

$$\dot{L}'_a(t) = \left(\vec{R} \times \vec{F}_{ext}\right)_a \quad \textbf{(642)}$$

As shown in this problem of the section on the equations of motion for many-particle systems, the right-hand side of this result of that is the total torque $\vec{N}(t)$, resulting in the second equation of movement of a rigid body, for the time derivative of the angular momentum

subs (*rhs* (**(642)**) = N_[a](t), **(642)**)

$$\dot{L}'_a(t) = \vec{N}_a(t) \quad \textbf{(643)}$$

Non-inertial coordinate systems

When describing the motion of a particle as seen from a non-inertial reference system (e.g. a rotating planet, like the Earth), we also see "acceleration" that is not due to any force but instead to the fact that the reference system itself is accelerated.

Problem

Consider a *non-inertial* reference system K' which moves with *non-constant* translational velocity $\vec{V}(t)$ with regards to an *inertial* reference system K_0.

a) Show that the Lagrangian L' in K' is given by

$$L' = \frac{1}{2}\, m\, \vec{v'} - m\, \vec{W} \cdot \vec{r'} - U$$

where $\vec{W} = \dot{\vec{V}}$ is the translational *acceleration* of the frame K' as seen from K_0.

b) Show that the Lagrange equation derived from this Lagrangian in the frame K' is

$$m\,\dot{\vec{v}}'(t) = -\nabla U - m\,\vec{W}$$

Solution

restart;
with(*Physics*:-*Vectors*) :

a) The starting point is the Lagrangian in the frame K_0. Denoting vectors and the Lagrangian in K_0 with the suffix 0, L_0 is given by

CompactDisplay$\left(\left(r_{0\_}, v_{0\_}, r'\_, v'\_, V\_, W\_\right)(t)\right)$

$$\vec{r}_0(t) \textit{ will now be displayed as } \vec{r}_0$$
$$\vec{v}_0(t) \textit{ will now be displayed as } \vec{v}_0$$
$$\vec{r}'(t) \textit{ will now be displayed as } \vec{r}'$$
$$\vec{v}'(t) \textit{ will now be displayed as } \vec{v}'$$
$$\vec{V}(t) \textit{ will now be displayed as } \vec{V}$$
$$\vec{W}(t) \textit{ will now be displayed as } \vec{W} \quad \textbf{(644)}$$

$L_0 = \frac{1}{2} m \cdot v_{0\_}(t)^2 - U\left(r_{0\_}(t)\right)$

$$L_0 = \frac{m\,\vec{v}_0^{\,2}}{2} - U\left(\vec{r}_0\right) \quad \textbf{(645)}$$

The velocities of the particle in the frames K_0 and K' are related by

$v_{0\_}(t) = v'\_(t) + V\_(t)$

$$\vec{v}_0 = \vec{v}' + \vec{V} \quad \textbf{(646)}$$

from which the Lagrangian L' is given by

subs$\left(\textbf{(646)}, U\left(r_{0\_}(t)\right) = U(r'\_(t)), L_0 = L', \textbf{(645)}\right)$

$$L' = \frac{m\left(\vec{v}' + \vec{V}\right)^2}{2} - U\left(\vec{r}'\right) \quad \textbf{(647)}$$

Note the use of L' in the above - in Maple 2023, when you load *Physics*, the prime does not represent differentiation, so that *primed variables* can be used to represent transformations, coordinates or vectors of different systems. Having primes represent differentiation can be restored via *Setup*(*primedvariables* = *false*).

expand(**(647)**)

$$L' = \frac{m\,\left\|\vec{v}'\right\|^2}{2} + m\left(\vec{v}' \cdot \vec{V}\right) + \frac{m\,\left\|\vec{V}\right\|^2}{2} - U\left(\vec{r}'\right) \quad \textbf{(648)}$$

In this result, the term $\frac{m\,\|\vec{V}\|^2}{2}$ does not depend on the coordinates $\vec{r}'$ or velocity $\vec{v}'$ and is a function of time t only. As such, it can be omitted from the Lagrangian

$subs\left(Norm(V\_(t))^2 = 0, \mathbf{(648)}\right)$

$$L' = \frac{m\,\|\vec{v}'\|^2}{2} + m\,(\vec{v}' \cdot \vec{V}) - U(\vec{r}') \qquad \mathbf{(649)}$$

The term $m\,(\vec{v}' \cdot \vec{V})$ can be rewritten as a total derivative plus a term involving the acceleration $\vec{W} = \dot{\vec{V}}$ of the frame K'. Starting from the product

$r'\_(t) \cdot V\_(t)$

$$\vec{r}' \cdot \vec{V} \qquad \mathbf{(650)}$$

$(\%diff = diff)\,(\mathbf{(650)}, t)$

$$\frac{\partial}{\partial t}\,(\vec{r}' \cdot \vec{V}) = \dot{\vec{r}}' \cdot \vec{V} + \vec{r}' \cdot \dot{\vec{V}} \qquad \mathbf{(651)}$$

$subs(diff(r'\_(t), t) = v'\_(t), diff(V\_(t), t) = W\_(t), \mathbf{(651)})$

$$\frac{\partial}{\partial t}\,(\vec{r}' \cdot \vec{V}) = \vec{v}' \cdot \vec{V} + \vec{r}' \cdot \vec{W} \qquad \mathbf{(652)}$$

$isolate(\mathbf{(652)}, v'\_(t) \cdot V\_(t))$

$$\vec{v}' \cdot \vec{V} = \frac{\partial}{\partial t}\,(\vec{r}' \cdot \vec{V}) - \vec{r}' \cdot \vec{W} \qquad \mathbf{(653)}$$

Introducing this into the Lagrangian L'

$subs(\mathbf{(653)}, \mathbf{(649)})$

$$L' = \frac{m\,\|\vec{v}'\|^2}{2} + m\left(\frac{\partial}{\partial t}\,(\vec{r}' \cdot \vec{V}) - \vec{r}' \cdot \vec{W}\right) - U(\vec{r}') \qquad \mathbf{(654)}$$

We get the expected result by discarding the total derivative $\frac{d}{dt}\left(\vec{r}' \cdot \vec{V}\right)$

$eval(\mathbf{(654)}, \%diff = 0)$

$$L' = \frac{m\,\|\vec{v}'\|^2}{2} - m\,(\vec{r}' \cdot \vec{W}) - U(\vec{r}') \qquad \mathbf{(655)}$$

---

b) The equations of motion can be computed from $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$

$\%diff(\%diff(\mathbf{(655)}, v'\_(t)), t) - \%diff(\mathbf{(655)}, r'\_(t))$

$$\frac{\partial}{\partial t}\,\frac{\partial}{\partial \vec{v}'}\left(L' = \frac{m\,\|\vec{v}'\|^2}{2} - m\,(\vec{r}' \cdot \vec{W}) - U(\vec{r}')\right) - \frac{\partial}{\partial \vec{r}'}\left(L' = \frac{m\,\|\vec{v}'\|^2}{2} - m\,(\vec{r}' \cdot \vec{W}) - U(\vec{r}')\right) \qquad \mathbf{(656)}$$

$value(\mathbf{(656)})$

$$0 = m\,\dot{\vec{v}}' + m\,\vec{W} + \mathrm{D}(U)\left(\vec{r}'\right) \quad \textbf{(657)}$$

where the last term $\mathrm{D}(U)\left(\vec{r}'\right)$ can be interpreted as the Gradient in the K' system.

Coriolis force and centripetal force

Problem

Consider a second *non-inertial* frame of reference J whose origin coincides with that of K', but which rotates relative to K' with variable angular velocity $\vec{\Omega}(t)$. Denote the position vector and velocity in the *non-inertial* frame J as $\vec{r}$ and $\vec{v}$,

a) Show that the Lagrangian L in the *non-inertial* frame J is given by

$$L = \frac{1}{2}\,m\,\vec{v}^{\,2} + m\,\vec{v}\cdot\left(\vec{\Omega}\times\vec{r}\right) + \frac{1}{2}\,m\,\left(\vec{\Omega}\times\vec{r}\right)^2 - m\cdot\left(\vec{W}\cdot\vec{r}\right) - U$$

b) Show that the Lagrange equation derived from this Lagrangian in the frame J is

$$m\,\dot{\vec{v}} = -\nabla U - m\,\vec{W} + m\,\left(\vec{r}\times\dot{\vec{\Omega}}\right) + 2\,m\,\left(\vec{v}\times\vec{\Omega}\right) + m\,\left(\vec{\Omega}\times\left(\vec{r}\times\vec{\Omega}\right)\right)$$

where $2\,m\,\left(\vec{v}\times\vec{\Omega}\right)$ is the Coriolis force and $m\,\left(\vec{\Omega}\times\left(\vec{r}\times\vec{\Omega}\right)\right)$ is the centrifugal force.

Solution

restart;
with(*Physics*:-*Vectors*) :

a) The starting point is the Lagrangian in the frame K_0. Denoting vectors and the Lagrangian in K_0 with the suffix 0, L_0 is given by

$CompactDisplay\left(\left(r_{0\_}, v_{0\_}, r'\_, v'\_, r\_, v\_, V\_, W\_, Omega\_\right)(t)\right)$

$\vec{r}_0(t)$ *will now be displayed as* $\vec{r}_0$

$\vec{v}_0(t)$ *will now be displayed as* $\vec{v}_0$

$\vec{r}'(t)$ *will now be displayed as* $\vec{r}'$

$\vec{v}'(t)$ *will now be displayed as* $\vec{v}'$

$\vec{r}(t)$ *will now be displayed as* $\vec{r}$

$\vec{v}(t)$ *will now be displayed as* $\vec{v}$

$\vec{V}(t)$ *will now be displayed as* $\vec{V}$

$\vec{W}(t)$ *will now be displayed as* $\vec{W}$

$\vec{\Omega}(t)$ *will now be displayed as* $\vec{\Omega}$ **(658)**

$$L_0 = \frac{1}{2} m\cdot v_{0\_}(t)^2 - U\left(r_{0\_}(t)\right)$$

$$L_0 = \frac{m \vec{v}_0^{\,2}}{2} - U(\vec{r}_0) \quad \textbf{(659)}$$

The velocities of the particle in the frames K_0 and K' are related by

$v_{0\_}(t) = v'\_(t) + V\_(t)$

$$\vec{v}_0 = \vec{v}' + \vec{V} \quad \textbf{(660)}$$

where $\vec{V}$ is the *translational* velocity of K' viewed from K_0. Inserting this relation **(660)** into L_0 gives L', the result of the previous problem

(655)

$$L' = \frac{m \|\vec{v}'\|^2}{2} - m\,(\vec{r}' \cdot \vec{W}) - U(\vec{r}') \quad \textbf{(661)}$$

In turn, the velocities $\vec{v}'$ and $\vec{v}$ in the frames K' and J are related by

$v'\_(t) = v\_(t) + Omega\_(t) \times r\_(t);$

$$\vec{v}' = \vec{v} + \vec{\Omega} \times \vec{r} \quad \textbf{(662)}$$

where $\vec{\Omega}$ is the *angular* velocity of the frame J viewed from K'. Also, since K' and J have the same origin, $\vec{r}' = \vec{r}$ and the Lagrangian L in J is

subs(**(662)**, $r'\_ = r\_$, $L' = L$, **(661)**)

$$L = \frac{m \|\vec{v} + \vec{\Omega} \times \vec{r}\|^2}{2} - m\,(\vec{r} \cdot \vec{W}) - U(\vec{r}) \quad \textbf{(663)}$$

expand(**(663)**)

$$L = \frac{m \|\vec{v}\|^2}{2} + m\,(\vec{v} \cdot (\vec{\Omega} \times \vec{r})) + \frac{m \|\vec{\Omega}\|^2 \|\vec{r}\|^2}{2} - \frac{m\,(\vec{\Omega} \cdot \vec{r})^2}{2} - m\,(\vec{r} \cdot \vec{W}) - U(\vec{r}) \quad \textbf{(664)}$$

Two of these terms can be regrouped as $(\vec{\Omega} \times \vec{r})^2$

$(Omega\_(t) \times r\_(t))^2$

$$(\vec{\Omega} \times \vec{r})^2 \quad \textbf{(665)}$$

expand(**(665)**) = **(665)**

$$\|\vec{\Omega}\|^2 \|\vec{r}\|^2 - (\vec{\Omega} \cdot \vec{r})^2 = (\vec{\Omega} \times \vec{r})^2 \quad \textbf{(666)}$$

simplify(**(664)**, {**(666)**})

$$L = \frac{m \|\vec{v}\|^2}{2} + m\,(\vec{v} \cdot (\vec{\Omega} \times \vec{r})) - m\,(\vec{r} \cdot \vec{W}) + \frac{m\,(\vec{\Omega} \times \vec{r})^2}{2} - U(\vec{r}) \quad \textbf{(667)}$$

which is the expected result.

---

b) To compute Lagrange's equation in vectorial form, one can use the standard formula as in the previous problem

%diff(*%diff*(**(667)**, $v\_(t)$), t) − *%diff*(**(667)**, $r\_(t)$)

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \vec{v}}\left(L = \frac{m\,\|\vec{v}\|^2}{2} + m\left(\vec{v}\cdot\left(\vec{\Omega}\times\vec{r}\right)\right) - m\left(\vec{r}\cdot\vec{W}\right) + \frac{m\left(\vec{\Omega}\times\vec{r}\right)^2}{2} - U(\vec{r})\right) - \frac{\partial}{\partial \vec{r}} \left(L = \frac{m\,\|\vec{v}\|^2}{2} + m\left(\vec{v}\cdot\left(\vec{\Omega}\times\vec{r}\right)\right) - m\left(\vec{r}\cdot\vec{W}\right) + \frac{m\left(\vec{\Omega}\times\vec{r}\right)^2}{2} - U(\vec{r})\right) \quad (668)$$

Evaluate these derivatives and replace $\dot{\vec{r}} = \vec{v}$

subs(*diff*(r_(t), t) = v_(t), *value*(**(668)**))

$$0 = m\,\dot{\vec{v}} + m\left(\dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times\vec{v}\right) - m\left(\vec{v}\times\vec{\Omega}\right) + m\,\vec{W} - m\left(\left(\vec{\Omega}\times\vec{r}\right)\times\vec{\Omega}\right) + \mathrm{D}(U)(\vec{r}) \quad (669)$$

Isolating $m\cdot\dot{\vec{v}}$ and collecting vector products we get the expected result

isolate(**(669)**, $m\cdot$*diff*(v_(t), t))

$$m\,\dot{\vec{v}} = -m\left(\dot{\vec{\Omega}}\times\vec{r} - \vec{v}\times\vec{\Omega}\right) + m\left(\vec{v}\times\vec{\Omega}\right) - m\,\vec{W} + m\left(\left(\vec{\Omega}\times\vec{r}\right)\times\vec{\Omega}\right) - \mathrm{D}(U)(\vec{r}) \quad (670)$$

collect(**(670)**, `&x`)

$$m\,\dot{\vec{v}} = m\left(\left(\vec{\Omega}\times\vec{r}\right)\times\vec{\Omega}\right) - m\left(\dot{\vec{\Omega}}\times\vec{r}\right) + 2\,m\left(\vec{v}\times\vec{\Omega}\right) - m\,\vec{W} - \mathrm{D}(U)(\vec{r}) \quad (671)$$

Part II (forthcoming)

The Hamiltonian and equations of motion; Poisson brackets

Canonical transformations

The Hamilton-Jacobi equation

References

References

[1] L.D. Landau, E.M. Lifshitz, **Mechanics**, **Course of Theoretical Physics**, Volume 1, third edition, Elsevier.
[2] Feynman, R.P.; Leighton, R.B.; and Sands, M. **The Feynman Lectures on Physics - Volume 1**, Addison-Wesley, 1977.