Constructing Minimal Triangular Mesh Based on Discrete Mean Curvature

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***Abstract*-In this paper, a so-called Plateau-Mesh problem is proposed, that is, to find a triangular mesh with the boundary formed by a given spacial closed polygon, whose area is minimum among all triangular meshes with the same boundary. To solve this problem, the former work by minimizing a function describing the mesh area directly, cannot obtain the global minimum of the function, only obtain its local minimum. In order to overcome this shortcoming, a new method to minimize the objective function which is measured by discrete mean curvatures is presented. As a base of the algorithm, the partial derivatives of discrete mean curvatures of the triangular mesh are evaluated. Numerical examples and error analysis are also given and the results show that our algorithm is correct and effective.**

Keywords-Plateau-Mesh problem; triangular mesh; least squares; discrete mean curvature; minimal surface

I. INTRODUCTION

A minimal surface, whose area is the minimum and geometric structure is steady, is the one for which mean curvatures everywhere vanish and Gauss curvatures are constant and negative everywhere except in finite points. A lot of people made researches on it in the past. Lagrange [1] gave the quasi-linear second-order elliptic partial differential equations for minimal surfaces, and Joseph Plateau found that the soap membrane made on the frame would become a minimal surface. Rado [2] and Douglas [3] obtained the solution for a disc minimal surface by the conformal mapping and the variational method respectively. Arnal, Lluch and Monterde [4] approximated minimal surfaces using Bézier triangular surfaces. However, all of these algorithms above cannot be applied directly to the engineering. The reason is that the solution of the minimal surface with a given boundary is just needed urgently. Such a class of minimal surface with the perfect shape and stress form is the most reasonable ideal initial state of the membrane structure, which is widely used in modern high-rise buildings, e.g., in the cable-membrane structure system of gymnasiums and airport lounges. In addition, this class of minimal surface plays an important role in the fuselage construction, hull manufacturing, molecular chemistry, crystallography and so on.

In 1993, Pinkall and Polthier [5] solved the problem of constructing minimal surfaces with the boundary of continuous or piecewise continuous curves. During 2003 and 2004, Monterde [6], [7] formally proposed the Plateau-Bézier problem: finding the Bézier surface whose area is the smallest in all the Bézier surfaces with the same boundary of a given spacial closed Bézier curve, i.e., the mean curvature everywhere on the obtained Bézier surface always vanishes. And Montverde's work took a key step towards the engineering applications for minimal surfaces.

In the last decade, some scholars devoted themselves to the discrete minimal surfaces researches. Dziuk and Hutchinson [8] approximated minimal surfaces by optimizing their discrete finite elements; Polthier and Rossman [9] gave some explicit expressions of the discrete catenoids and the discrete right helicoids using the variational method. But none of their methods above yet belong to the discrete Plateau method or Plateau-Mesh method.

In 2010 and 2011, Pan and Xu [17], [18] gave the solutions to the problem of the discrete minimal surface with the given boundary using different subdivision methods, which are very similar with discrete Plateau problem.

Plateau-Mesh problem can be defined as follows: find a triangular mesh whose geometric area is the smallest in all triangular meshes with the same boundary formed by a given spacial closed polygon. In 2008, Chen, et al [10] solved such a problem by optimizing the initial triangular mesh area. However, this algorithm obtained only the local minimum usually, but not necessarily the global minimum. To improve the above non-consummate method, we propose an algorithm of optimizing the discrete mean curvatures function to solve the Plateau-Mesh problem.

II. ALGORITHM

A. Algorithm Overview

Our algorithm for the Plateau-Mesh problem can be described by the following sentences briefly: firstly, picking up randomly the initial uncertain n points from the set of candidate points according to the method which will be shown in detail in the second paragraph; secondly, carrying out the Delauny triangulation with the boundary of a given spacial closed polygon L for them; thirdly, taking the discrete mean curvatures on the mesh got by the first step as the objective function and solve it using the least squares algorithm to get a new triangular mesh; then repeating the last two steps until that any of iterative conditions isn't satisfied. The discrete mean curvature on each vertex of the mesh can be defined by many ways, e.g., by the weighted edge by Taubin [11], by the discretization of LBO (Laplace-Beltrami Operator) operators by Desbrun et al [12], by bounding boxes by Cohen-Steiner et al [13]. Here we employ the last definition by Cohen-Steiner et al [13].

The method to select the initial set of candidate points can be introduced as follows. Set vertices of the spacial closed polygon L as $\{\boldsymbol{V}_i, i=1,\cdots,m\}$ and the candidate points set as $\{\boldsymbol{V}_j^{'}, j=1,\cdots,k, k\geq n\}$, respectively. The first candidate point is picked up by $\boldsymbol{V}_1^{'}=\frac{1}{m}\sum_{i=1}^{m}\boldsymbol{V}_i$ and the remaining candidate points are selected using the triangular barycentric formulas, i.e., $\boldsymbol{V}_j^{'}=\frac{1}{3}\left(\boldsymbol{V}_j+\boldsymbol{V}_{j-1}+\boldsymbol{V}_1^{'}\right), j=2,3,\cdots,m;$ $\boldsymbol{V}_{m+1}^{'}=\frac{1}{3}\left(\boldsymbol{V}_1+\boldsymbol{V}_2+\boldsymbol{V}_2^{'}\right),\ \ \boldsymbol{V}_{m+2}^{'}=\frac{1}{3}\left(\boldsymbol{V}_1+\boldsymbol{V}_1^{'}+\boldsymbol{V}_2^{'}\right),$ $\boldsymbol{V}_{m+3}^{'}=\frac{1}{3}\left(\boldsymbol{V}_2+\boldsymbol{V}_1^{'}+\boldsymbol{V}_2^{'}\right),\cdots,$ until $k\geq n.$ The Delaunay triangulation is carried out using the random incremental method [14]. Denote the discrete mean curvature for each vertex $\boldsymbol{V}_i$ on the mesh by $H(\boldsymbol{V}_i)$ and the objective function will be $F(\boldsymbol{V}_{m+1},\cdots,\boldsymbol{V}_{m+n})=\sum_{i=1}^{m+n}\left|H(\boldsymbol{V}_i)\right|^2$. Fig. 1 describes simply the algorithm process.

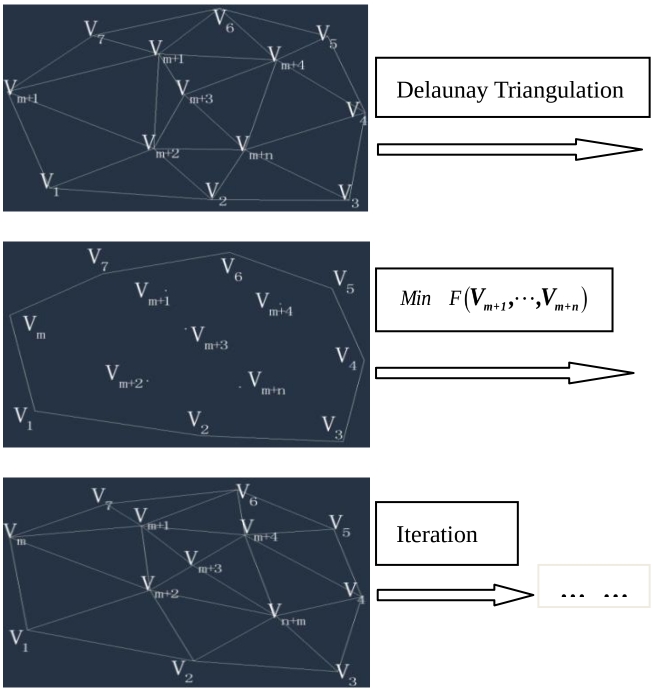

Fig. 1 The algorithm process for the Plateau-Mesh problem

B. The Formula of Discrete Mean Curvatures

Discrete mean curvatures on a mesh are defined by Cohen-Steiner et al [13] as follows,

$$H_i \equiv H(\boldsymbol{V}_i)\equiv\frac{1}{|B|}\sum_{e\in N_1(\boldsymbol{V}_i)}\beta(\boldsymbol{e})\,length(\boldsymbol{e}\cap B),\quad i=1,\cdots,m+n.$$

Where $|B|$ represents the geometric area of the region B that means the bound of the region in which the mean curvatures are measured; $\beta(\boldsymbol{e})$ represents the signed angle between the normal vectors of two triangles sharing the edge $\boldsymbol{e}$ (as shown in Fig. 4) and is positive for a convex crease or negative for a concave one; $N_1(\boldsymbol{V}_i)$ is the 1-neighborhood of the vertex $\boldsymbol{V}_i$; $length(\boldsymbol{e}\cap B)$ is the length of the portion of the edge $\boldsymbol{e}$ within the region B; $|B|$ can be chosen in many ways but here we choose it as the area of mass center of $N_1(\boldsymbol{V}_i)$, i.e., $|B|=\frac{1}{3}\sum_{f\in N_1(\boldsymbol{V}_i)}A_f$, where A_f represents the geometric area of the triangle f, as shown in Fig. 2.

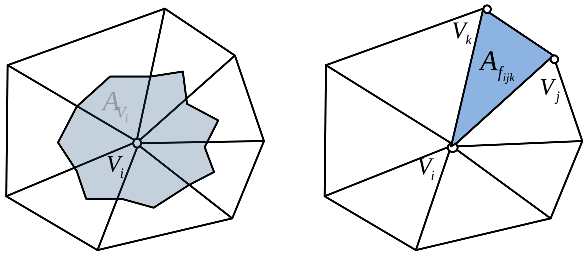

Fig. 2 Areas of a vertex and triangle

Then the objective function can be written by

$$F(\boldsymbol{x})=\frac{1}{2}\sum_{i=1}^{m+n}|H_i|^2,\ \boldsymbol{x}=(\boldsymbol{V}_{m+1},\cdots,\boldsymbol{V}_{m+n}). \qquad (1)$$

Minimal surfaces are the surfaces with the vanishing mean curvatures, so the discrete minimal mesh surfaces are the mesh surfaces with vanishing discrete mean curvatures. In the algorithm process partial derivatives of the discrete mean curvatures on vertices $\boldsymbol{V}_i,\ \ i=m+1,\cdots,m+n$ are needed to be evaluated.

C. Partial Derivatives of Discrete Mean Curvatures

Partial derivatives of discrete mean curvatures H_i on vertices $\boldsymbol{V}_j,\ \ j=m+1,\cdots,m+n$ are given as follows [15],

$$\begin{aligned}\frac{\partial}{\partial\boldsymbol{V}_j}H_i=&\frac{1}{|B|}\sum_{e\in N_1(\boldsymbol{V}_i)}\left[\left(\frac{\partial}{\partial\boldsymbol{V}_j}\beta(\boldsymbol{e})\right)\left|\frac{1}{2}\boldsymbol{e}\right|+\left(\frac{\partial}{\partial\boldsymbol{V}_j}\left|\frac{1}{2}\boldsymbol{e}\right|\right)\beta(\boldsymbol{e})\right]\\&-\frac{1}{|B|^2}\frac{\partial}{\partial\boldsymbol{V}_j}|B|\sum_{e\in N_1(\boldsymbol{V}_i)}\beta(\boldsymbol{e})\left|\frac{1}{2}\boldsymbol{e}\right|.\end{aligned}\qquad (2)$$

Then we evaluate three types of partial derivatives in the formula (2), respectively.

1) Partial Derivative of the Geometric Area $|B|$: The formula is given by [15]

$$\frac{\partial}{\partial \boldsymbol{V}_j}|B| = \frac{1}{3}\sum_{f\in N_1(\boldsymbol{V}_i)} \frac{\partial}{\partial \boldsymbol{V}_j} A_f .$$

As an example, the partial derivative of the triangle $V_1V_2V_3$'s area on the vertex $\boldsymbol{V}_1$ is evaluated by the following formula,

$$\frac{\partial}{\partial \boldsymbol{V}_1} A_{123} = \frac{1}{2}\cdot\frac{1}{|\boldsymbol{e}_{12}\times\boldsymbol{e}_{13}|}\cdot\left(-\boldsymbol{e}_{12}\cdot(\boldsymbol{e}_{23}\cdot\boldsymbol{e}_{13})+\boldsymbol{e}_{13}\cdot(\boldsymbol{e}_{23}\cdot\boldsymbol{e}_{12})\right)$$

$$=\frac{1}{2}\cdot\frac{1}{|\boldsymbol{e}_{12}\times\boldsymbol{e}_{13}|}\cdot\left((\boldsymbol{e}_{12}\times\boldsymbol{e}_{13})\times\boldsymbol{e}_{23}\right)=\frac{1}{2}\cdot(\bar{\boldsymbol{n}}\times\boldsymbol{e}_{23}),$$

as shown in Fig. 3, where $\bar{\boldsymbol{n}}$ is the unit normal of the triangle $V_1V_2V_3$ and $\boldsymbol{e}_{23}$ the opposite edge to the vertex $\boldsymbol{V}_1$

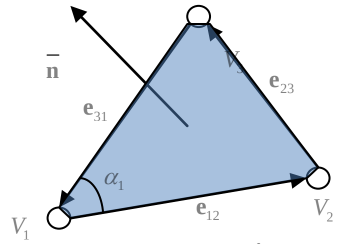

Fig. 3 Meanings of signs

2) Partial Derivative of the Edge Length $|\boldsymbol{e}_{12}|$: The formula is given by [15]

$$\frac{\partial}{\partial \boldsymbol{V}_1}|\boldsymbol{e}_{12}| = -\bar{\boldsymbol{e}}_{12}, \qquad \frac{\partial}{\partial \boldsymbol{V}_2}|\boldsymbol{e}_{12}| = \bar{\boldsymbol{e}}_{12},$$

also as shown in Fig. 3, where $\bar{\boldsymbol{e}}_{12}$ is the unit vector in the direction of the edge vector $\boldsymbol{e}_{12}$.

3) Partial Derivative of the Angle $\beta(\boldsymbol{e})$: The formula is given by

$$\frac{\partial}{\partial \boldsymbol{V}_3}\beta(\boldsymbol{e}) = \mathbf{sign}(\beta(\boldsymbol{e}))\cdot\frac{1}{|\boldsymbol{n}_1\times\boldsymbol{n}_2|}\cdot\left((\boldsymbol{n}_1\cdot\boldsymbol{n}_2)\cdot\left(\frac{(\boldsymbol{n}_1\times\boldsymbol{e}_{41})}{|\boldsymbol{n}_2|^2}+\frac{(\boldsymbol{n}_2\times\boldsymbol{e}_{24})}{|\boldsymbol{n}_1|^2}\right)-(\boldsymbol{n}_2\times\boldsymbol{e}_{41}+\boldsymbol{n}_1\times\boldsymbol{e}_{24})\right). \quad (3)$$

Where $\text{sign}(\beta(\boldsymbol{e}))$ is defined as follows and the meanings of signs are shown in Fig. 4.

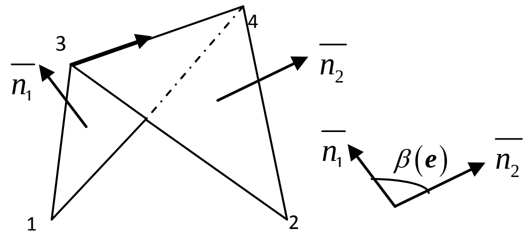

Fig. 4 The signed angle between the normal vectors of two triangles incident to $\boldsymbol{e}$

$$\text{sign}(\beta(\boldsymbol{e})) = \begin{cases}1, \beta(\boldsymbol{e})\ge 0\\ -1, \beta(\boldsymbol{e})<0\end{cases}, \quad \boldsymbol{n}_1 = \boldsymbol{e}_{13}\times\boldsymbol{e}_{14}, \boldsymbol{n}_2 = \boldsymbol{e}_{24}\times\boldsymbol{e}_{23}.$$

Here we give the proof of the formula (3). By evaluating the partial derivatives for $\boldsymbol{V}_3$ in both sides of the following equation, $\cos\beta(\boldsymbol{e}) = \bar{\boldsymbol{n}}_1\cdot\bar{\boldsymbol{n}}_2 = \frac{\boldsymbol{n}_1\cdot\boldsymbol{n}_2}{|\boldsymbol{n}_1|\cdot|\boldsymbol{n}_2|}$, we obtain

$$-\sin\beta(\boldsymbol{e})\cdot\frac{\partial}{\partial \boldsymbol{V}_3}\beta(\boldsymbol{e}) = \frac{1}{|\boldsymbol{n}_1|\cdot|\boldsymbol{n}_2|}\cdot\frac{\partial}{\partial \boldsymbol{V}_3}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2) - \frac{\boldsymbol{n}_1\cdot\boldsymbol{n}_2}{(|\boldsymbol{n}_1|\cdot|\boldsymbol{n}_2|)^2}\frac{\partial}{\partial \boldsymbol{V}_3}(|\boldsymbol{n}_1|\cdot|\boldsymbol{n}_2|). \quad (4)$$

The partial derivative of the inner product of $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, which are the normal vectors of the two triangles, can be evaluated by

$$\frac{\partial}{\partial \boldsymbol{V}_3}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2) = \frac{\partial}{\partial \boldsymbol{V}_3}\left((\boldsymbol{e}_{13}\times\boldsymbol{e}_{14})\cdot(\boldsymbol{e}_{24}\times\boldsymbol{e}_{23})\right)$$

$$= \frac{\partial}{\partial \boldsymbol{V}_3}\left((\boldsymbol{e}_{13}\cdot\boldsymbol{e}_{24})\cdot(\boldsymbol{e}_{14}\cdot\boldsymbol{e}_{23})-(\boldsymbol{e}_{13}\cdot\boldsymbol{e}_{23})\cdot(\boldsymbol{e}_{14}\cdot\boldsymbol{e}_{24})\right)$$

$$= (\boldsymbol{e}_{14}\cdot\boldsymbol{e}_{23})\cdot\boldsymbol{e}_{24}+(\boldsymbol{e}_{13}\cdot\boldsymbol{e}_{24})\cdot\boldsymbol{e}_{14}-(\boldsymbol{e}_{14}\cdot\boldsymbol{e}_{24})\cdot\boldsymbol{e}_{23}-(\boldsymbol{e}_{14}\cdot\boldsymbol{e}_{24})\cdot\boldsymbol{e}_{13}$$

$$= \boldsymbol{e}_{14}\times(\boldsymbol{e}_{24}\times\boldsymbol{e}_{23}) - \boldsymbol{e}_{24}\times(\boldsymbol{e}_{13}\times\boldsymbol{e}_{14})$$

$$= \boldsymbol{n}_2\times\boldsymbol{e}_{14}+\boldsymbol{n}_1\times\boldsymbol{e}_{24}.$$

The partial derivative of the product of $|\boldsymbol{n}_1|$ and $|\boldsymbol{n}_2|$, which are the lengths of the above two normal vectors, can be evaluated by

$$\frac{\partial}{\partial \boldsymbol{V}_3}(|\boldsymbol{n}_1|\cdot|\boldsymbol{n}_2|) = 4\cdot\frac{\partial}{\partial \boldsymbol{V}_3}(A_{341}\cdot A_{324})$$

$$= 4\cdot\frac{1}{2}\cdot(\bar{\boldsymbol{n}}_1\times\boldsymbol{e}_{41})\cdot A_{324} + 4\cdot\frac{1}{2}\cdot(\bar{\boldsymbol{n}}_2\times\boldsymbol{e}_{24})\cdot A_{341}$$

$$= (\boldsymbol{n}_1\times\boldsymbol{e}_{41})\cdot\frac{|\boldsymbol{n}_2|}{|\boldsymbol{n}_1|}+(\boldsymbol{n}_2\times\boldsymbol{e}_{24})\cdot\frac{|\boldsymbol{n}_1|}{|\boldsymbol{n}_2|}.$$

Substituting the following three equations, $\frac{\partial}{\partial \boldsymbol{V}_3}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2) = \boldsymbol{n}_2\times\boldsymbol{e}_{14}+\boldsymbol{n}_1\times\boldsymbol{e}_{24}$, $\frac{\partial}{\partial \boldsymbol{V}_3}(|\boldsymbol{n}_1|\cdot|\boldsymbol{n}_2|) = (\boldsymbol{n}_1\times\boldsymbol{e}_{41})\cdot\frac{|\boldsymbol{n}_2|}{|\boldsymbol{n}_1|}+(\boldsymbol{n}_2\times\boldsymbol{e}_{24})\cdot\frac{|\boldsymbol{n}_1|}{|\boldsymbol{n}_2|}$ and $\sin\theta = \frac{|\boldsymbol{n}_1\times\boldsymbol{n}_2|}{|\boldsymbol{n}_1|\cdot|\boldsymbol{n}_2|}$ into Eq. (4), it's easy to obtain the formula (3).

D. The Least Squares Algorithm: Levenberg-Marquardt (LM)

This section will introduce the least squares Levenberg-Marquardt (LM) [16] algorithm which is employed to minimize the objective function written as the formula (1). In fact, LM algorithm is an improvement of Gauss-Newton algorithm.

The objective function (1) satisfies $F(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{f}(\boldsymbol{x})^{\boldsymbol{T}}\cdot\boldsymbol{f}(\boldsymbol{x})$, where $\boldsymbol{f}(\boldsymbol{x})\equiv(H_1,H_2,\cdots,H_{m+n})$ is the vector function, so minimizing $F(\boldsymbol{x})$ is equivalent to minimizing $\|f(x)\|_{L_2}$. Also the Jacobin matrix of the vector function $f(x)$ is $J=\left(\frac{\partial H_i}{\partial V_j}\right)_{(m+n)*n}$.

In the $(k+1)th$ iterative step, x_{k+1} is updated by the following formula,

$$\left(J^T J+\mu_k I\right)\delta_k=-J^T f(x_k),\quad x_{k+1}=x_k+\delta_k.$$

Where μ_k is the damping parameter, whose initial value is assigned to the 10^{-6} times of the largest principal diagonal element of the matrix $J^T J$ and the updating function is given by literature [14] or reviewed in detail in the following pseudo code (as shown in Fig. 5). The termination criterion of the iteration in LM algorithm can be given as

$$|F_k-F_{k-1}|<\varepsilon(1+F_k),\quad \|\nabla F_k\|_\infty<\sqrt[3]{\varepsilon}(1+F_k),$$
$$\|\delta_k\|_\infty<\sqrt{\varepsilon}\left(1+\|\delta_k\|_\infty\right),\quad k_{\max}<k.$$

Where ε is assigned to 10^{-6} and $L(\delta)$ is defined by $F(x)+\delta^T J^T f+\frac{1}{2}\delta^T J^T J\delta$ as shown in Fig. 5.

III. Examples and Error Analysis

This section will show some examples and give error analysis of our algorithm. The examples include three well-known types of minimal surfaces, i.e., Spiral surfaces, Saddle surfaces and Scherk surfaces, as well as the simulative membrane structure of the engineering.

In the case of the three well-known types of minimal surfaces, with the spacial closed polygon L given respectively by the three types of minimal surfaces approximately as the triangular mesh boundary, we solve the Plateau-Mesh problem by the algorithm and process error analysis for the final discrete solution compared with the corresponding continuous surface. In the first two examples six figures are given, where Fig.-s. 6(1) and 7(1) are the initial Delaunay triangular meshes; Fig.-s 6(2) and 7(2) are the corresponding final discrete solutions; Fig.-s 6(3) and 7(3) are corresponding comparisons evaluated by the Chen et al [10] `s algorithm; Fig.-s 6(4) and 7(4) are plotted by the corresponding error analysis from different perspectives, where large or small errors are denoted by dense or sparse sets of points. Histogram Fig. 8 is plotted by nine sets of the third type of minimal surfaces experimental data.

In the case of the simulative membrane structure of the engineering, with the spacial closed polygon L given by the steady points of the simulation membrane structure as the triangular mesh boundary, we solve the Plateau-Mesh problem and Fig. 9 is plotted according to the discrete solution, where Fig. 9(1) is the initial Delaunay triangular mesh and Fig. 9(2) the final discrete solution.

A. Helicoid

LM algorithm

Input: space closed polylines L and initial points x_0

Output: the solution x of the *Plateau-Mesh* problem

begin

$k:=0;\quad v:=2;\quad x:=x_0$

$A:=J(x)^T J(x);\quad g:=J(x)^T f(x)$

$found:=\|\nabla F(x)\|_\infty<\sqrt[3]{\varepsilon}\left(1+F(x)\right)$

$\mu:=10^{-6}\cdot\max\{a_{ii}\}$

While $(!found)$ *and* $(k<k_{\max})$

$k:=k+1;$ **Solve** $(A+\mu I)\delta=-g$

if $\left(\|\delta\|_\infty<\sqrt{\varepsilon}\left(1+\|\delta\|_\infty\right)\right)$ $found=\text{true}$

else

$x_{new}=x+\delta$

$\rho=\left(F(x)-F(x_{new})\right)/\left(L(0)-L(\delta)\right)$

if $\left(F(x)-F(x_{new})<\varepsilon\left(1+F(x)\right)\right)$

$found=\textbf{true}$

if $(\rho>0\,\&!\,found)$

$x=x_{new}$

$\mathbf{A}=\mathbf{J}(x)^T\mathbf{J}(x);\quad \mathbf{g}=\mathbf{J}(x)^T\mathbf{f}(x)$

if $\left(\|\nabla F(x)\|_\infty<\sqrt[3]{\varepsilon}\left(1+F(x)\right)\right)$

$found=\textbf{true}$

$\mu=\mu*\max\left\{\frac{1}{3},1-(2\rho-1)^3\right\};\quad v=2$

else

$\mu=\mu*v;\quad v=2*v$

end

Fig. 5 The pseudo code of the LM algorithm

The surface equation $r(u,v)=(u\cdot\cos v,u\cdot\sin v,b\cdot v)$, where $b=1.9$, $n=20$ and the number of the vertices of the polygon L is 28. The results are shown by Fig.-s 6(1), 6(2) and 6(4) and the comparison solved by the Chen et al [10] `s algorithm is shown in Fig. 6(3); our algorithm`s mean error is 2.941e-003 and the Chen et al [10] `s is 4.110e-003, which are both evaluated by the following formula,

$$ERR:=\begin{cases}\left|y-x\tan(z/b)\right|\cdot\left(1/|y|\right), & |y|>1;\\ \left|y-x\tan(z/b)\right|, & \text{other.}\end{cases}$$

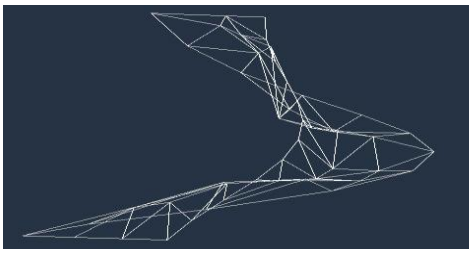

Fig. 6(1) Initial Delaunay triangular mesh based on the helicoids

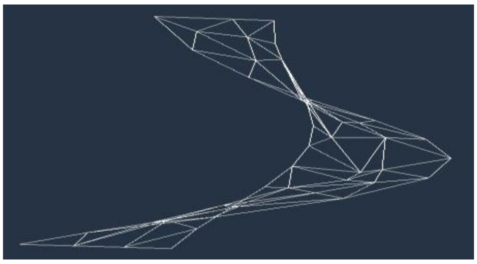

Fig. 6(2) Discrete solution by our algorithm

Fig. 6(3) Discrete solution by the Chen et al [10] `s algorithm

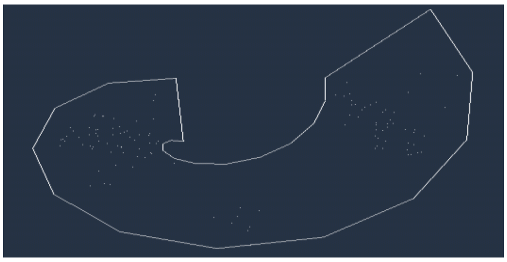

Fig. 6(4) Error analysis from different perspectives, large or small errors denoted by dense or sparse sets of points

B. Catenoid

The equation $\boldsymbol{r}(u,v)=(b\cos u\cosh v, bv, b\sin u\cosh v)$, where $b=2.05$, $n=25$ and the number of the vertices of the polygon L is 24. The results are shown by Fig.-s 7(1), 7(2) and 7(4) and the comparison solved by the Chen et al [10] `s algorithm is shown in Fig. 7(3); our algorithm`s mean error is 9.192e-004 and the Chen et al [10] `s is 1.412e-003, which are both evaluated by the following formula,

$$ERR := \begin{cases} \left|z-\sqrt{b^2\cosh^2 y/b - x^2}\right|, & |z|<1, \\ \left|z-\sqrt{b^2\cosh^2 y/b - x^2}\right| \Big/ |z|, & \text{other}. \end{cases}$$

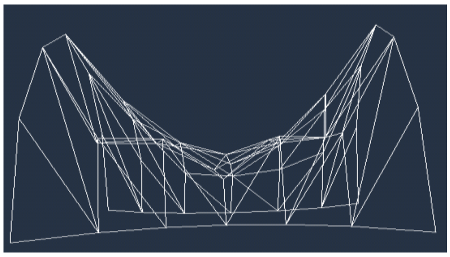

Fig. 7(1) Initial Delaunay triangular mesh based on the catenoids

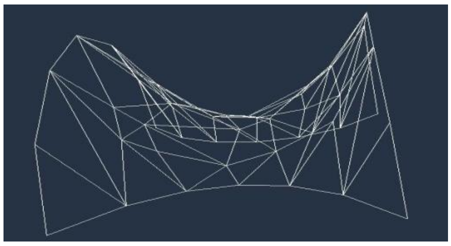

Fig. 7(2) Discrete solution by our algorithm

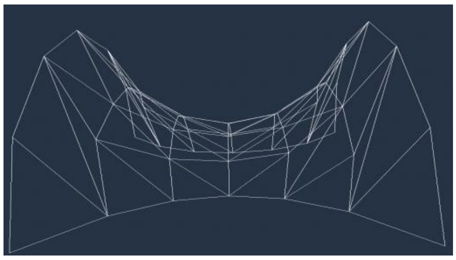

Fig. 7(3) Discrete solution by the Chen et al [10] `s algorithm

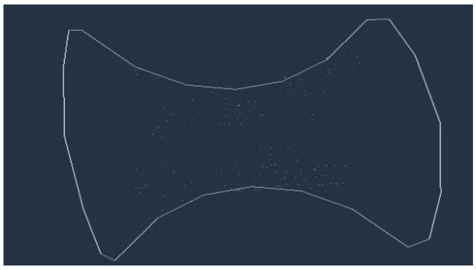

Fig. 7(3) Error analysis from different perspectives, large or small errors denoted by dense or sparse sets of points

C. Scherk

The surface equation $z=\dfrac{1}{b}\cdot\ln\left(\cos(by)/\cos(bx)\right)$, where $b=0.4,\ 1,\ 4.9$ with different n and different numbers of the vertices of the polygon L . Histogram Fig. 8 is plotted to show mean errors, in which the horizontal axis indicates serial numbers of examples and the vertical axis indicates mean errors whose formulas are defined by

$$ERR := \begin{cases} \left|z-1/b\cdot\ln\left(\cos(by)/\cos(bx)\right)\right|, & |z|<1, \\ \left|z-1/b\cdot\ln\left(\cos(by)/\cos(bx)\right)\right| \Big/ |z|, & \text{other}. \end{cases}$$

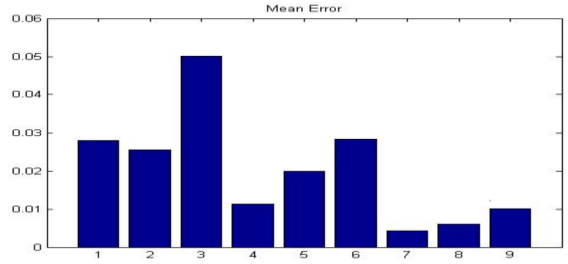

Fig. 8 Histogram of error analysis based on the scherks

D. Application in Enginerring

Simulative membrane structures of the engineering, where $n = 4$ and the numbers of the vertices of the polygon L is 9. The result is shown by Fig.-s 9(1) and 9(2) and the mean error of mean curvatures is 4.786e-004.

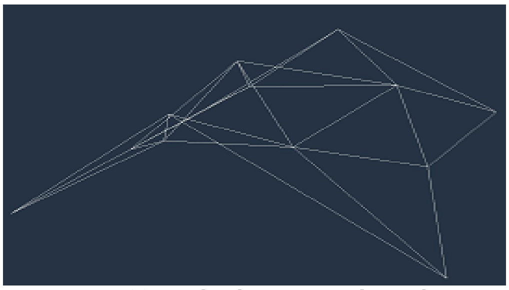

Fig. 9(1) Initial Delaunay triangular mesh

Fig. 9(2) discrete solution correspondingly

IV. CONCLUSIONS

For the triangular Plateau-Mesh problem, in 2008 Chen et al [10] optimized the initial triangular mesh geometric area to solve it, but their solution were not necessarily the global minimum; we optimize the mean curvatures defined by Cohen-Steiner et al [13] of the initial triangular mesh and our algorithm`s advantage is more obvious for relative sparse set of points. The algorithm is proved to be correct and effective by numerical examples. But there are still some weaknesses, e.g., error accuracy needs to be improved; the initial uncertain points would influence the solution. In the future the relationship in theory between the two algorithms: one based on the geometric area and the other based on discrete curvatures, need to be researched further.

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