Exploration of prime producing trinomial f(n)=n^2+n+41
by Matthew C. Anderson

Abstract

In this study, of the trinomial, f(n)=n^2+n+41, assume n is a positive integer. I used Maple, computer algebra system, to calculate whether f(n) is a prime number, or a composite number, for n less than 1800. Observe many parabolas in the data of the graph. It turns out that the curve fit is exact.

In number theory, assume n is a positive integer. Let

$f(n)=n^2+n+41.$ (expression 1)

It was shown by Legendre, in 1798 that if $0 \le n \le 40$ then f(n) is a prime number. Certain patterns become evident when considering points (a,n) where

$f(n) \equiv 0$ modulo a. (expression 2)

The collection of all such points, up to the limit 1800, we are calling a "graph of discrete divisors". This graph has been, analytically, curve fit, exactly, with parabolas. The parabolas are described by closed form expressions. The parabolas are indexed (r,c) by pairs of relatively prime positive integers. The expressions for the middle parabolas are

$p(r,c) = (c*x - r*y)^2 - x*(c*x - r*y) - x + 41*r^2$ (expression 3)

The restrictions on p(r,c) are that $0<r<c$ and gcd(r,c) = 1, where the greatest common divisor of two arguments is written gcd(r,c). And all four of r, c, x, and y are integers.

When we take the derivative of p(r,c) with respect to x and set this expression equal to zero, we obtain

$x = (163*m^2)/4$ (expression 4)

where m is the x minimum of a given parabola

Each such pair (r,c) yields (again determined by curve fit and by observation of "graph of discrete divisors".) And using Maple, computer algebra system for the coefficients of the parabolas, and there is one parabola per pair (r,c). Calculations of parametric integer polynomials $a*z^2 + b*z + c$, where the coefficients (a, b, and c) are determined for each parabola, which pass through data points in this "graph of discrete divisors". The first few (r,c) pairs are (2,1); (3,2); (3,1); (4,3); (4,1) and (5,4). Again, r and c must be relatively prime numbers. Further, the quartic $f(a*z^2 + b*z + c)$ will factor algebraically over the integers into two quadratic expressions. We call this our “parabola conjecture”. Certain structure of the “graph of discrete divisors” are due to elementary relationships between pairs of co-prime integers.

We conjecture that all composite values of f(n) arise by substituting integer values of z into $f(a*z^2 + b*z + c)$, where this quartic polynomial factors algebraically over **Z** for $a*z^2 + b*z + c$, which is a quadratic polynomial determined by a pair of relatively prime integers (r,c). We are confident of this conjecture because of numerical evidence and the structure of the “graph of discrete divisors” produced by some computer code in our computer algebra system (Maple). We call this our “no stray points conjecture” because all the points in the graph appear to lie on a parabola.

We further conjecture that the minimum x-values for parabolas corresponding to (r,c) are given by expression 4. The vertical lines are given by $x = 163*m^2/4$ where m = 2, 3, 4, … The numerical evidence seems to support this. This is called our “parabolas line up conjecture”.

Theorem 1 – Consider f(n) with n a positive integer. Then f(n) never has an integer factor less than 41.

We prove Theorem 1 with a modular construction. We make a residue table of f(y) modulo x, with all the prime number divisors less than 41. A form of the fundamental theorem of arithmetic states that any integer greater than one is either a prime number, or can be written as a unique product of prime numbers (ignoring the order of numbers). So if our residue table never shows a prime factor less than 41, then by extension, f(n) never has a prime factor less than 41.

For example, to determine that f(n) is never divisible by 2, note the first column of the residue table. If n is even, then f(n) is odd. Similarly, if n is odd, then f(n) is also odd. In either case, f(n) does not have factorization by 2. Since all integers are either even or odd, f(n) is never divisible by 2, when n is a positive integer.

Also, for divisibility by 3, there are 3 cases to check. They are $n \equiv 0$, 1, and 2 modulo 3. f(0) modulo 3 is 2. f(1) modulo 3 is 1 and f(2) modulo 3 is 2. Since none of these results is 0, we have that f(n) is never divisible by 3. This is the second column of the residue table.

The number 0 is first found in the residue table for the cases f(0) modulo 41 and f(40) modulo 41. We can see that $40^2 + 40 + 41 = 41^2$. This means that if n is congruent to 0 mod 41 then f(n) will be divisible by 41. What’s more is that these are the only two cases for divisibility by 41. Similarly, if n is congruent to 40 modulo 41 then f(n) will also be divisible by 41.

After the residue table, we observe a curve fit to our “graph of discrete divisors”, which has points when f(y) modulo x is divisible by x. This is a perfect curve fit. The points (x,y) can be seen in a data table, and on the graph.

Thus, we have shown that f(n) never has an integer factor less than 41.

Theorem 2

Since f(a) = a^2 + a + 41, we want to show that f(a) = f(-a-1).

Proof of theorem 2
Because q(a) = a*(a+1) + 41,
Now q(-a-1) = (-a-1)*(-a-1+1) + 41.
So, q(-a-1) = (-a-1)*(-a) + 41,
And f(-a-1) = f(a).
End of proof of theorem 2.

Corollary 1
Further, if f(b) modulo c ≡ 0 then q(c-b-1) modulo c ≡ 0

We see that it is amazing that the data points all fall within an exact curve fit. All the parabolas have integer coefficients.

End first section

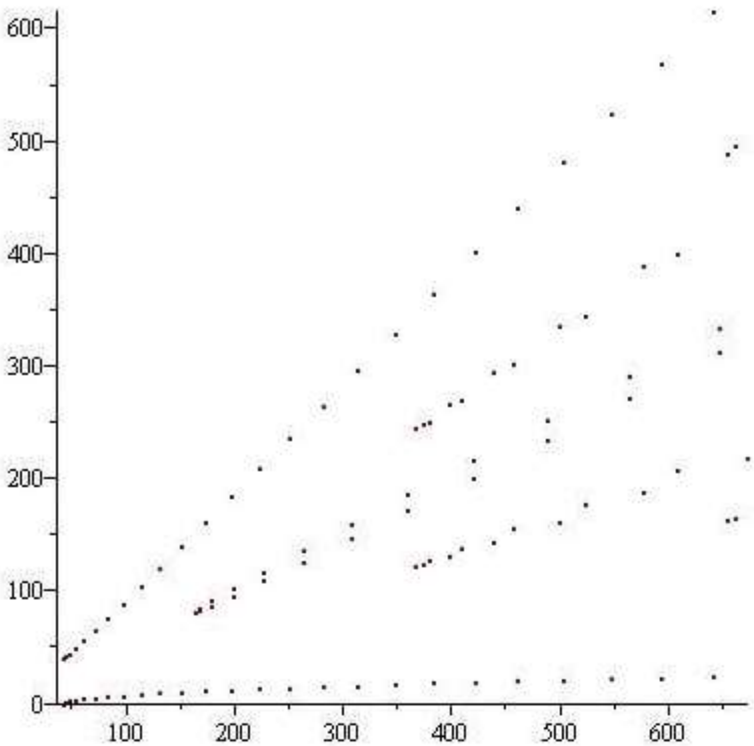
This residue table shows divisibility restrictions on f(n).

note that the number 0 does not appear in most columns.

Residue Table

	2	3	5	7	11	13	17	19	23	29	31	37	41	43
0	1	2	1	6	8	2	7	3	18	12	10	4	0	41
1	1	1	3	1	10	4	9	5	20	14	12	6	2	0
2		2	2	5	3	8	13	9	1	18	16	10	6	4
3			3	4	9	1	2	15	7	24	22	16	12	10
4			1	5	6	9	10	4	15	3	30	24	20	18
5				1	5	6	3	14	2	13	9	34	30	28
6				6	6	5	15	7	14	25	21	9	1	40
7					9	6	12	2	5	10	4	23	15	11
8					3	9	11	18	21	26	20	2	31	27
9					10	1	12	17	16	15	7	20	8	2
10					8	8	15	18	13	6	27	3	28	22
11						4	3	2	12	28	18	25	9	1
12						2	10	7	13	23	11	12	33	25
13							2	14	16	20	6	1	18	8
14							13	4	21	19	3	29	5	36
15							9	15	5	20	2	22	35	23
16							7	9	14	23	3	17	26	12
17								5	2	28	6	14	19	3
18								3	15	6	11	13	14	39
19									7	15	18	14	11	34
20									1	26	27	17	10	31
21									20	10	7	22	11	30
22									18	25	20	29	14	31
23										13	4	1	19	34
24										3	21	12	26	39
25										24	9	25	35	3
26										18	30	3	5	12
27										14	22	20	18	23
28										12	16	2	33	36
29											12	23	9	8
30											10	9	28	25
31												34	8	1
32												24	31	22
33												16	15	2
34												10	1	27
35												6	30	11
36												4	20	40
37													12	28
38													6	18
39													2	10
40													0	4
41														0
42														41

Note that h(y) = f(y) = y^2+y+41.
Also, this “bifurcation graph” is also called a graph of discrete divisors.

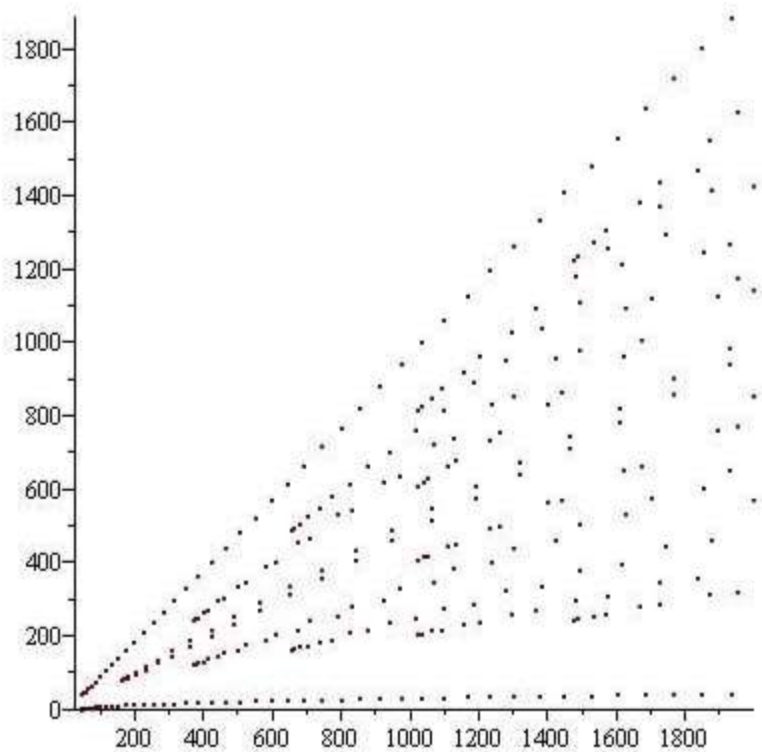


Bifurcation Graph

These are pairs of numbers (x,y) such that h(y) mod x ≡ 0.

And h(y) = $y^2 + y + 41$.

Similar graph, zoomed out some



Here is a zoomed out iteration of the same graph as the previous page.

There seems to b an apparent regular structure in this graph of divisibility.

The points give themselves to an exact curve fit of parabolas.

Note that all the points in this interesting graph fit on a given parabola.

No stray points

The points give themselves to an exact curve fit of parabolas.

The general form of these parabolas is –

$p(r,c) = c^2x^2 - 2c*r*x*y + r^2y^2 - (c*r+1)*x + r^2y + 41r^2$. (Equation 1).

p is for parabola, r is for row index, c is for column index, x is the horizontal axis and y is the vertical axis.

This does not include the top and bottom parabolas.

There are also 3 restrictions.

$1<r$

$0<c<r$

Gcd(r,c) = 1.

All the parabolas can be described exactly and algebraically.

Excuse the white space

The x minimum of p(r,c) is

Pmin = (163*r^2)/4. (expression 2)

This can be found with the Mape Commmand extrema.

To wit –

```
> # p is for parabola
> p[r, c] := c^2·x^2 − r·c·2·x·y + r^2·y^2 − (r·c + 1)·x + r^2y + 41·r^2;
```

$$p_{r,c} := c^2 x^2 - 2\,r\,c\,x\,y + r^2 y^2 - (c\,r + 1)\,x + r^2 y + 41\,r^2 \qquad (1)$$

```
> e2 := extrema(x, p[r, c] = 0, {x, y});
```

$$e2 := \left\{\frac{163}{4}\,r^2\right\} \qquad (2)$$

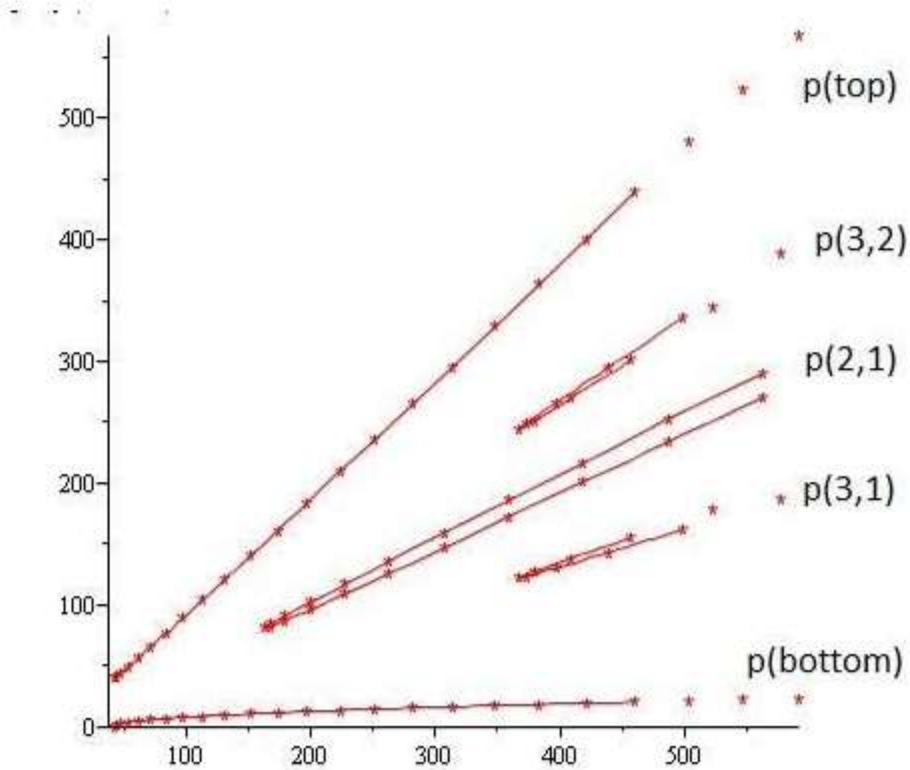
That is good to know.
Maple is a useful tool.

Here is some Maple code to show the exact curve fit for the graph of divisors.

```
> # Maple code
> x[bottom] := z^2+z+41; y[bottom] := z;
> p2 := plot([x[bottom], y[bottom], z = 0 .. 20]);
> with(plots);
> x[1, 1, top] := z^2+z+41; y[top] := z^2+40;
> p3 := plot([x[top], y[top], z = 0 .. 20]);
>
> y[2, 1] := 2*z^2+z+81; x[2, 1] := 4*z^2+163;
> p4 := plot([x[2, 1], y[2, 1], z = -10 .. 10]);
>
> y[3, 1] := 3*z^2+2*z+122; x[3, 1] := 9*z^2+3*z+367;
> p5 := plot([x[3, 1], y[3, 1], z = -4 .. 3]);
>
> y[3, 2] := 6*z^2+z+244; x[3, 2] := 9*z^2+3*z+367;
> p6 := plot([x[3, 2], y[3, 2], z = -4 .. 3]);
>
> d1 := display([p2, p3, p4, p5, p6])
> # code for graph of divisors
> xv := Vector[row](89); yv := Vector[row](89); counter := 1;
> for a from 2 to 600 do
    for b from 0 to a-1 do
      if `mod`(b^2+b+41, a) = 0 then
        xv[counter] := a; yv[counter] := b; counter := counter+1
      end if
    end do
  end do;
> counter;
> d2 := plot(xv, yv, style = point, symbol = asterisk);
> display(d1, d2)
> # This produces a graph.
```

This curve fit took some effort. Some manual, some computer aided

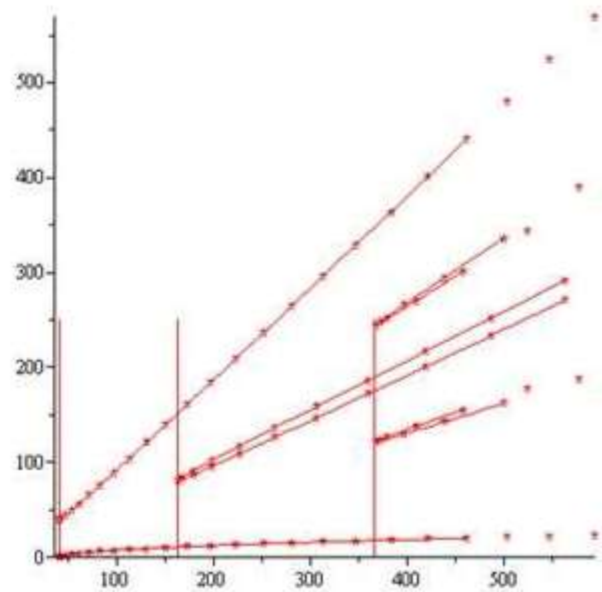
Graph of Divisors with parabolas that exactly fit the points



Pretty graph

Graph of divisibility

Vertical lines at
163*c^2/4



Notice the vertical lines are tangent to the parabolas.

Glad you are still reading

Undiscovered Expressions

So far, we want to determine when $h(n) = n^2 + n + 41$ is a prime number. We produce a dataset that satisfies the congruency $h(y) \equiv 0 \bmod x$. In other words, we find ordered pairs (x,y) such that x divides h(y). The graph of all pairs (x,y) seems to have obvious regularity and patterns. We are able to tabulate coefficients of parabolas that exactly fit the data. Here are the first few parabolas :

P bottom x (z) = $z^2 + z + 41$
P bottom y (z) = z

P top x (z) = $z^2 - z + 41$
P top y (z) = $z^2 + 40$

P 2,1 x (z) = $4*z^2 + 163$
P 2,1 y (z) = $2*z^2 + z + 81$

P 3,2 x (z) = $4*z^2 + 163$
P 3,2 y(z) = $6*z^2 + z + 244$

P 3,1 x (z) = $z^2 + z + 41$
P 3,1 y (z) = $3*z^2 + 2*z + 122$

A computer tool can show that h(P 2,1 x(z)) = P 2,1 y(z) * $(z^2 + z + 41)$. (equation *)

The Maple command subs() can substitute one expression into another. Also the Maple command factor() can factor quartic polynomials.

The important part of equation * is that the right hand side is the product of two integers, both greater than one. This proves that h(P 2,1(z)) is a composite number. In other words, if you put a positive integer of the form $4*z^2 +163$ as input to h(n), then you will get a composite number as output.

We have the general parabola

P c,r x(z) and P c,r y(z).

I was unable to determine these expressions. It may be impossible and it is related to the distribution of prime numbers.

My naming scheme for the parabolas requires c and r to be integers and

0>r>c and gcd(r,c) = 1

Where gcd is the Greatest Common Divisor of two integers.

That is a lot to read.

Appendix 1 - Maple Code for graph of discrete divisors

```
x := Vector(55) :
y := Vector(55) :
counter := 1 :
for a from 2 to 378 do
for b from 0 to a − 1 do
if mod(b^2 + b + 41, a) = 0
  then x[counter] := a : y[counter] := b : counter := counter + 1;
end if;
end do:
end do:
```

The number 378 was chosen by trial and error to completely fill the vector of length 55. The number 55 was chosen so that we can easily identify 5 parabolas from the data points.

This code creates a data set and stores it in two vectors.

```
> # list of pairs (x,y) such that y^2 + y + 41 mod x ≡ 0.
> for a from 1 to 40 do
  x[a], y[a]
  end do;
```

41, 0
41, 40
43, 1
43, 41
47, 2
47, 44
53, 3
53, 49
61, 4
61, 56
71, 5
71, 65
83, 6
83, 76
97, 7
97, 89
113, 8
113, 104
131, 9
131, 121
151, 10
151, 140
163, 81
167, 82
167, 84

It is interesting. There are patterns, man.

```
> h := n^2 + n + 41 :
> f := proc(y)
    description "factors the substitution of the eypression into n^2+n + 41";
    factor(y^2 + y + 41);
    end proc;
```

$f :=$ **proc**(y) **(1)**

description "factors the substitution of the eypression into n^2+n + 41";

factor(y^2 + y + 41)

end proc

\>

\> *# Small equation coeffieients doublecheck*

\>

\> *#The question I am attempting to answer in this project is − what integer values of n cause h(n)* **to** *be a composite number,* **and by** *extention, when is h(n) prime.*

\> *# r is for row and c is for column. So y[r,c] is a composition of functions h(y[r,c]).*

\> *# when y[r,c] is carefully chosen, it makes y[r,c] algebraically. This means that y[r,c] is the product of two integers, neither of which is 1 or −1, and thus y[r, c] is composite*

\> *# I am pretty sure that any n below a threshold lies on one of the lines described by the expressions below.*

\>

\>

\> $y[1, 1] := z:$
$x[1, 1] := f(\%);$

$$x_{1,\,1} := z^2 + z + 41 \qquad \textbf{(2)}$$

\> $y[1, 2] := z^2 + 40:$
$x[1, 2] := f(\%);$

$$x_{1,\,2} := (z^2 + z + 41)\,(z^2 - z + 41) \qquad \textbf{(3)}$$

\> $y[2, 1] := 2\,z^2 + z + 81:$
$x[2, 1] := f(\%);$

$$x_{2,\,1} := (4\,z^2 + 163)\,(z^2 + z + 41) \qquad \textbf{(4)}$$

\> $y[3, 1] := 3\,z^2 + 2\,z + 122:$
$x[3, 1] := f(\%);$

$$x_{3,\,1} := (z^2 + z + 41)\,(9\,z^2 + 3\,z + 367) \qquad \textbf{(5)}$$

\> $y[3, 2] := 6\,z^2 + z + 244:$
$x[3, 2] := f(\%);$

$$x_{3,\,2} := (4\,z^2 + 163)\,(9\,z^2 + 3\,z + 367) \qquad \textbf{(6)}$$

\> $y[4, 1] := 4\,z^2 + 3\,z + 163:$
$x[4, 1] := f(\%);$

$$x \;:= (16\,z^2 + 8\,z + 653)\,(z^2 + z + 41) \qquad \textbf{(7)}$$

This represents efforts.

But enough about
$f(n) = n^2+n+41$.

Let us look at a new trinomial

n^2+n+17.

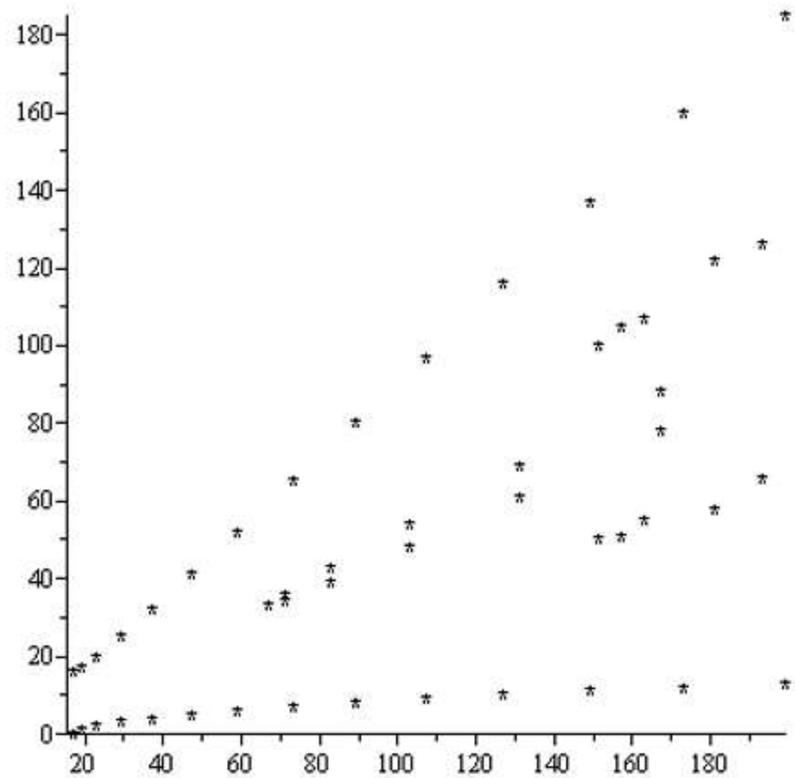
Analysis of the trinomial $f(n) = n^2 + n + 17$.

Abstract – Assuming that n is an non-negative integer, we find a pattern of when $f(n) = n^2 + n + 17$ is a composite number. We assign n as n = A*x^2+B*x+C. Where A, B, and C are determined by numerical evidence. The f(n) factors algebraically, and f(n) is a composite number.

We use the Maple program to calculate the values of 'n' where f(n) is a composite number. Then we graph these results. The graph shows some structure for the composite cases. See Maple code.

```
> # 6 - 29 - 2023
>
   x := Vector[row](49) :
   y := Vector[row](49) :
   counter := 1 :
   for a from 2 to 200 do
   for b from 0 to a - 1 do
   if mod(b^2 + b + 17, a) = 0
    then x[counter] := a : y[counter] := b : counter := counter + 1;
   end if;
   end do:
   end do:
> counter
> plot(x, y, style = point, symbol = asterisk, color = black )
```

Okay



> *# this is a graph of 49 data points of* $y^2 + y + 17 \ \mathbf{mod}\ x = 0$.

> *# It can be curve fit with parabolas.*

> *# This graph shows 5 parabolas*

> *# The names of the parabolas are* $p_{top}; p_{bottom}; p_{2,1}; p_{3,2}$; **and** $p_{3,1}$

.

>

Hope you find this page interesting.

Good fun

Citations

1) Elements of Number Theory by John Stillwell from Undergraduate Texts in Mathematics with Springer (2003) exercise 1.1.1

2) Prime-Producing Quadratics by R.A. Mollin American Mathematical Monthly, Vol 104 No. 6(Jun.-Jul. 1997) p529-544 http://links.jstor.org

3) Prime and Composite Polynomials by J.F. Ritt Transactions of the American Mathematical Society, Vol 23 No 1 (Jan., 1922) p 51-66 http://www.jstor.org/stable/1988911/

4) Polynomials with Large Numbers of Prime Values by Betty Garrison The American Mathematical Monthly, Vol 97, No . 4 (Apr., 1990), pp 316 - 317 http://www.jstor.org/stable/2324515/

5) Gauss' Class Number Problem for Imaginary Quadratic Fields by Dorian GoldFeld, Bulletin (New Series) of the American Mathematical Society Volume 13, Number 1, July 1985.

6) Weisstein, Eric W. "Prime-Generating Polynomial." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Prime-GeneratingPolynomial.html

7) Pegg, Ed Jr. "Bouniakowsky Conjecture" From Mathworld- A Wolfram Web Resource, created by Eric W. Weisstein http://mathworld.wolfram.com/BouniakowskyConjecture.html

8) Weisstein, Eric W. "Lucky Numbers of Euler." From Mathworld- A Wolfram Web Resource http://mathworld.wolfram.com/LuckyNumbersofEuler.html

9) Pegg, Ed Jr. "Bouniakowsky Conjecture." From Mathworld - A Wolfram Web Resource created by Eric W. Weisstein.

10) Weisstein, Eric W. "Landau's Problems." From Mathworld - A Wolfram Web Resource. http://mathworld.wolfram.com/LandausProblems.html see problem #4

Have a good day.