Classical–Quantum Approximation
for a Harmonic Oscillator Coupled to a Classical Projectile

Quantum Evolution: Time-Dependent Hamiltonian

The oscillator sees a time-dependent potential $V\big(x, y(\tau)\big)$. Its Schrödinger equation is

$$i\,\frac{\partial}{\partial\tau}\,\psi(x,\tau) = \Big[-\frac{1}{2m_x}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\,m_x x^2 + V\big(x, y(\tau)\big)\Big]\psi(x,\tau).$$

Expanding in the eigenbasis $\{\phi_n(x)\}$ with energies E_n gives

$$\psi(x,\tau) = \sum_{n=0}^{N-1} C_n(\tau)\,e^{-iE_n\tau}\,\phi_n(x).$$

Projecting onto ϕ_m yields

$$i\,\dot{C}_m(\tau) = \sum_{n=0}^{N-1} V_{mn}(\tau)\;e^{-i(E_n-E_m)\tau}\;C_n(\tau),$$

where

$$V_{mn}(\tau) = \langle\phi_m\,|\,V(x,y(\tau))\,|\,\phi_n\rangle = \lambda\,e^{-y(\tau)}\,M_{mn}.$$

Definition of $B_{mn}(\tau)$

It is often convenient to absorb the phase into a single coupling matrix,

$$B_{mn}(\tau) = e^{\,i\,(E_m-E_n)\,\tau}\;M_{mn},$$

so that the quantum–amplitude equations become

$$i\,\dot{C}_m(\tau) = \lambda\,e^{-y(\tau)}\sum_{n=0}^{N-1} e^{-\,i(E_n-E_m)\,\tau}\,M_{mn}\,C_n(\tau) = \lambda\,e^{-y(\tau)}\sum_{n=0}^{N-1} B_{mn}(\tau)\,C_n(\tau).$$

Analytic Form of M_{mn}

The static overlap matrix

$$M_{mn} = \int_{-\infty}^{\infty}\phi_m(x)\,e^{x}\,\phi_n(x)\,dx$$

can be expressed in closed form as

$$M_{mn} = \exp\!\Big(\tfrac{1}{4m_x}\Big)\,\sqrt{\frac{\min(m,n)!}{\max(m,n)!}}\,\Big(\tfrac{1}{\sqrt{2m_x}}\Big)^{|m-n|}\,L_{\min(m,n)}^{(|m-n|)}\!\Big(-\tfrac{1}{2m_x}\Big),$$

where $L_k^{(\alpha)}$ is the associated Laguerre polynomial.

Classical (Ehrenfest) Equations

The classical projectile evolves under the quantum expectation of the coupling:

$$\begin{aligned} \dot{y}(\tau) &= \frac{p_y(\tau)}{m_y}, \\ \dot{p}_y(\tau) &= -\frac{\partial}{\partial y}\Big\langle \psi(\tau) \Big| V(x,y) \Big| \psi(\tau) \Big\rangle = \lambda\, e^{-y(\tau)} \left\langle e^{x} \right\rangle_{\psi(\tau)}, \end{aligned}$$

with

$$\left\langle e^{x} \right\rangle = \sum_{m,n} C_m^*(\tau)\, B_{mn}(\tau)\, C_n(\tau).$$

Coupled ODE System

Putting both pieces together, one solves simultaneously:

$$\begin{aligned} i\,\dot{\mathbf{C}}(\tau) &= \lambda\, e^{-y(\tau)}\, B(\tau)\, \mathbf{C}(\tau), \\ \dot{y}(\tau) &= \frac{p_y(\tau)}{m_y}, \\ \dot{p}_y(\tau) &= \lambda\, e^{-y(\tau)}\, \mathbf{C}(\tau)^\dagger B(\tau)\, \mathbf{C}(\tau), \end{aligned}$$

where $\mathbf{C} = (C_0, \ldots, C_{N-1})^T$ and $B_{mn}(\tau)$ is defined above.