To simplify the expression under the assumptions

$$\delta_2 > 0, \quad \delta_2 > gA_c, \quad \delta_1 = 0,$$

we will work step-by-step with the complex square root expressions:

Given expressions:

$$e_1 = \sqrt{i\delta_2 \pm \sqrt{-A_c^2 g + (i\delta_2)^2}}, \quad e_2 = \sqrt{i\delta_2 \pm \sqrt{-A_c^2 g + (-i\delta_2)^2}}$$

Since $\delta_1 = 0$, these reduce to:

$$e_1 = \sqrt{i\delta_2 \pm \sqrt{-A_c^2 g - \delta_2^2}}, \quad e_2 = \sqrt{i\delta_2 \pm \sqrt{-A_c^2 g - \delta_2^2}}$$

Step 1: Define

Let us define a common quantity:

$$R := \sqrt{-A_c^2 g - \delta_2^2}$$

Because both $A_c^2 g > 0$ and $\delta_2^2 > 0$, the term inside the square root is **negative real**, so:

$$R = \sqrt{-(A_c^2 g + \delta_2^2)} = i\sqrt{A_c^2 g + \delta_2^2}$$

Step 2: Substitute R into the expressions

Now rewrite:

$$e_1 = \sqrt{i\delta_2 \pm i\sqrt{A_c^2 g + \delta_2^2}}, \quad e_2 = \sqrt{i\delta_2 \pm i\sqrt{A_c^2 g + \delta_2^2}}$$

Factor out i from each term inside the square root:

$$e_1 = \sqrt{i\left(\delta_2 \pm \sqrt{A_c^2 g + \delta_2^2}\right)}, \quad e_2 = \sqrt{i\left(\delta_2 \pm \sqrt{A_c^2 g + \delta_2^2}\right)}$$

Step 3: Express in exponential or polar form (optional)

Let us denote:

$$S = \delta_2 \pm \sqrt{A_c^2 g + \delta_2^2} > 0$$

Then:

$$e = \sqrt{iS} = \sqrt{S e^{i\pi/2}} = \sqrt{S} e^{i\pi/4} = \frac{\sqrt{2S}}{2}(1+i)$$

Thus:

$$e_1 = e_2 = \frac{\sqrt{2(\delta_2 \pm \sqrt{A_c^2 g + \delta_2^2})}}{2}(1+i)$$

Final Result

Both e_1 and e_2 become **complex expressions with equal real and imaginary parts** under the assumptions:

$$e_{1,2}^{(\pm)} = \frac{\sqrt{2(\delta_2 \pm \sqrt{A_c^2 g + \delta_2^2})}}{2}(1+i)$$

This form is useful for identifying oscillatory behavior, especially if these roots appear in wave solutions like $e^{e_{1,2}x}$.

Great. Let's now **approximate** the expression

$$\sqrt{\delta_2 \pm \sqrt{A_c^2 g + \delta_2^2}}$$

using the **binomial series**, under the assumption that $\delta_2 > gA_c$, so that the ratio

$$\varepsilon = \frac{A_c^2 g}{\delta_2^2} \ll 1$$

is a small quantity.

Step-by-step binomial approximation

Step 1: Expand the inner square root

We begin with:

$$\sqrt{A_c^2 g + \delta_2^2} = \delta_2 \sqrt{1+\varepsilon}$$

Use the binomial series expansion:

$$\sqrt{1+\varepsilon} \approx 1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \cdots$$

So:

$$\sqrt{A_c^2 g + \delta_2^2} \approx \delta_2 \left(1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2\right)$$

Step 2: Plug back into the outer square root

We now consider the two branches:

Positive branch:

$$\begin{aligned}
\sqrt{\delta_2 + \sqrt{A_c^2 g + \delta_2^2}} &\approx \sqrt{\delta_2 + \delta_2\left(1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2\right)} \\
&= \sqrt{2\delta_2 + \delta_2\left(\frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2\right)} \\
&= \sqrt{\delta_2\left(2 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2\right)} \\
&\approx \sqrt{2\delta_2}\left(1 + \frac{1}{8}\varepsilon - \frac{1}{32}\varepsilon^2\right)
\end{aligned}$$

Negative branch:

$$\begin{aligned}
\sqrt{\delta_2 - \sqrt{A_c^2 g + \delta_2^2}} &\approx \sqrt{\delta_2 - \delta_2\left(1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2\right)} \\
&= \sqrt{-\delta_2\left(\frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2\right)}
\end{aligned}$$

Since this is negative inside the root, the result is imaginary:

$$= i\sqrt{\delta_2\left(\frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2\right)}$$

Final Approximations

Using $\varepsilon = \frac{A_c^2 g}{\delta_2^2}$, we have:

- **Positive branch**:

$$\sqrt{\delta_2 + \sqrt{A_c^2 g + \delta_2^2}} \approx \sqrt{2\delta_2}\left(1 + \frac{A_c^2 g}{8\delta_2^2} - \frac{(A_c^2 g)^2}{32\delta_2^4}\right)$$

- **Negative branch**:

$$\sqrt{\delta_2 - \sqrt{A_c^2 g + \delta_2^2}} \approx i\sqrt{\frac{A_c^2 g}{2\delta_2}\left(1 - \frac{A_c^2 g}{4\delta_2^2}\right)}$$