SLIDING WITH FRICTION

ROUBEN ROSTAMIAN

A bead of mass m slides along the parametrically defined curve $\boldsymbol{r}(s) = \big\langle x(s), y(s)\big\rangle$ within a vertical plane. The motion is affected by dry friction with friction coefficient μ. We wish to find the equation of motion.

We take the x axis as horizontal and pointing to the right, while the y axis is vertical and points up. We write $\boldsymbol{\tau}(s)$ and $\boldsymbol{\nu}(s)$ for the unit tangent and unit normal to the curve. We have

$$\boldsymbol{\tau}(s) = \frac{1}{\big\|\boldsymbol{r}'(s)\big\|}\boldsymbol{r}'(s) = \frac{1}{\sqrt{x'(s)^2 + y(s)'^2}}\big\langle x'(s), y'(s)\big\rangle, \tag{1a}$$

$$\boldsymbol{\nu}(s) = \frac{1}{\sqrt{x'(s)^2 + y(s)'^2}}\big\langle -y'(s), x'(s)\big\rangle, \tag{1b}$$

where a prime indicates the derivative with respect to the parameter s.

The bead's position at any time t corresponds to a parameter value $s(t)$. The velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ are

$$\boldsymbol{v}(t) = \frac{d}{dt}\boldsymbol{r}\big(s(t)\big) = \boldsymbol{r}'\big(s(t)\big)\dot{s}(t), \tag{2a}$$

$$\boldsymbol{a}(t) = \frac{d}{dt}\boldsymbol{v}(t) = \boldsymbol{r}''\big(s(t)\big)\dot{s}(t)^2 + \boldsymbol{r}'\big(s(t)\big)\ddot{s}(t), \tag{2b}$$

where a dot indicates the derivative with respect to the time t.

Forces acting on the bead are of two kinds: (a) the weight $\boldsymbol{w} = m\langle 0, -g\rangle$, and (b) the constraint applied by the curve, which in the absence of friction would have been normal to the curve, as in $R\boldsymbol{\nu}$, but the presence of friction adds a tangential force $\pm\mu R\boldsymbol{\tau}$. The force always acts opposite the motion. Thus, if the velocity points in the $\boldsymbol{\tau}$ direction, the the force is $-\mu R\boldsymbol{\tau}$, and if the velocity points opposite to the $\boldsymbol{\tau}$ direction, the the force is $+\mu R\boldsymbol{\tau}$. Thus we express that force as

$$-\mu R(t)\,\boldsymbol{\tau}\big(s(t)\big)\,\operatorname{sign}\Big(\boldsymbol{\tau}\big(s(t)\big)\cdot\boldsymbol{v}(t)\Big),$$

where sign is the *signum function* defined as

$$\operatorname{sign}(x) = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

We conclude that the equation of motion is

$$m\,\boldsymbol{a}(t) = m\,\boldsymbol{g} + R(t)\,\boldsymbol{\nu}\big(s(t)\big) - \mu R(t)\,\boldsymbol{\tau}\big(s(t)\big)\,\operatorname{sign}\Big(\boldsymbol{\tau}\big(s(t)\big)\cdot\boldsymbol{v}(t)\Big), \tag{3}$$

We project the equation of motion onto the directions $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$, and thus resolve it into a set of two differential equations. Considering that $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are orthonormal,

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Date: August 24, 2026.

the equations are

$$m\,\boldsymbol{a}(t)\cdot\boldsymbol{\tau}\big(s(t)\big) = m\,\boldsymbol{g}\cdot\boldsymbol{\tau}\big(s(t)\big) - \mu R(t)\,\mathrm{sign}\Big(\boldsymbol{\tau}\big(s(t)\big)\cdot\boldsymbol{v}(t)\Big), \tag{4a}$$

$$m\,\boldsymbol{a}(t)\cdot\boldsymbol{\nu}\big(s(t)\big) = m\,\boldsymbol{g}\cdot\boldsymbol{\nu}\big(s(t)\big) + R(t). \tag{4b}$$

We simplify these equations as follows. From (1a) we have $\boldsymbol{r}'(s) = \big\|\boldsymbol{r}'(s)\big\|\,\boldsymbol{\tau}(s)$, therefore (2b) may be written as

$$\boldsymbol{a}(t) = \boldsymbol{r}''\big(s(t)\big)\dot{s}(t)^2 + \big\|\boldsymbol{r}'\big(s(t)\big)\big\|\,\boldsymbol{\tau}\big(s(t)\big)\ddot{s}(t).$$

Dot-multiplying this by $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$, and recalling the orthonormality of $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$, we see that

$$\boldsymbol{a}(t)\cdot\boldsymbol{\tau}\big(s(t)\big) = \boldsymbol{r}''\big(s(t)\big)\cdot\boldsymbol{\tau}\big(s(t)\big)\,\dot{s}(t)^2 + \big\|\boldsymbol{r}'\big(s(t)\big)\big\|\,\ddot{s}(t). \tag{5a}$$

$$\boldsymbol{a}(t)\cdot\boldsymbol{\nu}\big(s(t)\big) = \boldsymbol{r}''\big(s(t)\big)\cdot\boldsymbol{\nu}\big(s(t)\big)\,\dot{s}(t)^2, \tag{5b}$$

and thus, the equations (4) take the form

(6a)

$$\boldsymbol{r}''\big(s(t)\big)\cdot\boldsymbol{\tau}\big(s(t)\big)\,\dot{s}(t)^2 + \big\|\boldsymbol{r}'\big(s(t)\big)\big\|\,\ddot{s}(t) = \boldsymbol{g}\cdot\boldsymbol{\tau}\big(s(t)\big) - \frac{1}{m}\mu R(t)\,\mathrm{sign}\Big(\boldsymbol{\tau}\big(s(t)\big)\cdot\boldsymbol{v}(t)\Big),$$

(6b)

$$\boldsymbol{r}''\big(s(t)\big)\cdot\boldsymbol{\nu}\big(s(t)\big)\,\dot{s}(t)^2 = \boldsymbol{g}\cdot\boldsymbol{\nu}\big(s(t)\big) + \frac{1}{m}R(t).$$

We eliminate $R(t)$ between these two and arrive at the following second order ODE in the unknown $s(t)$:

$$\begin{aligned}\big\|\boldsymbol{r}'\big(s(t)\big)\big\|\,\ddot{s}(t) + \boldsymbol{r}''\big(s(t)\big)\cdot\Big(\boldsymbol{\tau}\big(s(t)\big) + Q(t)\,\boldsymbol{\nu}\big(s(t)\big)\Big)\,\dot{s}(t)^2 \\ - \boldsymbol{g}\cdot\Big(\boldsymbol{\tau}\big(s(t)\big) + Q(t)\,\boldsymbol{\nu}\big(s(t)\big)\Big) = 0,\end{aligned} \tag{7}$$

where we have set

$$Q(t) = \mu\,\mathrm{sign}\Big(\boldsymbol{\tau}\big(s(t)\big)\cdot\boldsymbol{v}(t)\Big).$$

Observe that the motion is independent of the mass m.